TORSION OF QUASI-ISOMORPHISMS

JAE-WOOK CHUNG AND XIAO-SONG LIN

ABSTRACT. In this paper, we introduce the notion of Reidemeister torsion for quasi-isomorphisms of based chain complexes over a field. We call a chain map a quasi-isomorphism if its induced homomorphism between homology is an isomorphism. Our notion of torsion generalizes the torsion of acyclic based chain complexes, and is a chain homotopy invariant on the collection of all quasi-isomorphisms from a based chain complex to another. It shares nice properties with torsion of acyclic based chain complexes, like multiplicativity and duality. We will further generalize our torsion to quasi-isomorphisms between free chain complexes over a ring under some mild condition. We anticipate that the study of torsion of quasi-isomorphisms will be fruitful in many directions, and in particular, in the study of links in 3-manifolds.

1. INTRODUCTION

The vector spaces used here are finite dimensional and rings are commutative and have $1 \neq 0$.

It was observed by Milnor in his beautiful paper "Infinite cyclic coverings" [4] that the Alexander polynomial of a knot and the Reidemeistor torsion of the infinite cyclic covering space of the knot complement is directly related to each other, because of the fact that the infinite cyclic covering of a knot complement is acyclic when tensoring with the field of rational functions.

Turaev generalized this theorem of Milnor to the case of links directly (see "Reidemeister torsion in knot theory" [3]). But in the case of links, since the maximal abelian covering space of a link complement can not be made acyclic in general, the statement of Turaev's generalization is not as nice as that of the theorem of Milnor.

One observation is that if we fix a link in S^3, say the trivial link L_0, then there are infinitely many links L in S^3, which admit natural maps $S^3 \setminus L \longrightarrow S^3 \setminus L_0$. These natural maps, when lifted to the corresponding maximal abelian covering spaces, will induce isomorphisms on homology after tensoring with the quotient field of the polynomial ring over $\mathbb{Z}$. Can we extend the notion of torsion to such a setting? If so, what will be the relationship between such torsion defined in this setting and the Alexander polynomial of L?

The second named author is supported in part by NSF.

The goal of this paper is then to offer an approach for possibly answering these questions. We introduce the notion of Reidemeister torsion for quasi-isomorphisms of based chain complexes over a field. We call a chain map a quasi-isomorphism if its induced homomorphism between homology is an isomorphism. Our notion of torsion generalizes the torsion of acyclic based chain complexes, and is a chain homotopy invariant on the collection of all quasi-isomorphisms from a based chain complex to another. It shares nice properties with torsion of acyclic based chain complexes, like multiplicativity and duality. We will further generalize our torsion to quasi-isomorphisms between free chain complexes over a ring under some mild condition. Since the acyclic condition is crucial whenever the notion of torsion is studied, we hope that our point of view of torsion would be useful in other directions.

The materials presented here contain only the most basic definitions and proofs of basic properties of torsion of quasi-isomorphisms. Although it was our original intention, we have not yet worked out the details of the application of our theory to the study of links in 3-manifolds. And the paper is written in a somehow unsophisticated way, trying to cover as much elementary details as possible. We hope that the reader will tolerate us. The reader may also notice that our exposition follows quite closely from that of the materials in Chapter 1 of Turaev's book [2]. Beside the introduction of the notion of torsion of quasi-isomorphism, there are many technical details in generalizing properties of torsion of acyclic chain complexes to that of quasi-isomorphisms. We consider these as the main contributions of this paper.

2. Basic definitions and preliminaries

In this section, we introduce basic terminologies and properties used in this paper.

Let V be a finite dimensional vector space over a field F. Suppose that $\dim_F V = n$ and $b = (b_1, \dots, b_n)$ and $b' = (b'_1, \dots, b'_n)$ are ordered bases for V. For convenience, let us use row vectors. Then for each $i \in \{1, \dots, n\}$, there is a unique $(a_{i1}, \dots, a_{in}) \in F^n$ such that $b_i = \sum_{j=1}^n a_{ij} b'_j$, hence, we have the transition matrix $(a_{ij})_{i,j=1,\dots,n}$ from b to b', denoted by (b/b'), which is an $n \times n$ invertible matrix and write $[b/b'] = \det(a_{ij})$.

Proposition 2.1. *Define $\sim$ on the set B of all ordered bases for a finite dimensional vector space V over a field F by $b \sim b'$ if and only if $[b/b'] = 1$ for all $b, b' \in B$. Then $\sim$ is an equivalence relation on B. We call ordered bases b and b' equivalent if $b \sim b'$.*

Proof. For each $b \in B$, (b/b) is the identity matrix, so $[b/b] = 1$. That is, $b \sim b$. If $b \sim b'$ in B, then $[b/b'] = 1$. Since $(b'/b) = (b/b')^{-1}$, $[b'/b] = 1$. Hence, $b' \sim b$. If $b \sim b'$ and $b' \sim b''$ in B, then $[b/b'] = 1$ and $[b'/b''] = 1$. We claim that $(b/b'') = (b/b')(b'/b'')$. For each $v \in V$, $v_{b'} = v_b(b/b')$ and $v_{b''} = v_{b'}(b'/b'')$. We have $v_{b''} = v_b(b/b')(b'/b'')$. Hence, $(b/b'') = (b/b')(b'/b'')$, so $[b/b''] = 1$. Therefore, $b \sim b''$. □

Lemma 2.2. *If a, b, c are ordered bases for a finite dimensional vector space V over a field F, then*

$$a \sim b \Leftrightarrow [a/c] = [b/c] \Leftrightarrow [c/a] = [c/b].$$

Proof. Notice that $[a/c] = [a/b][b/c]$. If $a \sim b$, then $[a/b] = 1$, so $[a/c] = [b/c]$. If $[a/c] = [b/c]$, then $[a/b] = 1$, so $a \sim b$. Also, since $[c/a] = [a/c]^{-1}$ and $[c/b] = [b/c]^{-1}$, we have $[a/c] = [b/c] \Leftrightarrow [c/a] = [c/b]$. □

Proposition 2.3. *If F is a field and $A \in M_m(F)$, $B \in M_n(F)$, $C \in M_{m\times n}(F)$, and $D \in M_{n\times m}(F)$, then*

$$\det \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det \begin{pmatrix} A & 0 \\ D & B \end{pmatrix} = \det A \, \det B.$$

Proof. By elementary row operations, we can change A and B to upper triangular matrices A' and B', respectively, so that $\det A = (-1)^r \det A'$ and $\det B = (-1)^s \det B'$ for some nonnegative integers r and s. Hence, we have

$$\begin{aligned} \det \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} &= (-1)^r(-1)^s \det \begin{pmatrix} A' & C' \\ 0 & B' \end{pmatrix} = (-1)^r(-1)^s \det A' \, \det B' \\ &= (-1)^r(-1)^s((-1)^r \det A)((-1)^s \det B) = \det A \, \det B \end{aligned}$$

for some $C' \in M_{m\times n}(F)$. Also, we have

$$\det \begin{pmatrix} A & 0 \\ D & B \end{pmatrix} = \det \begin{pmatrix} A & 0 \\ D & B \end{pmatrix}^t = \det \begin{pmatrix} A^t & D^t \\ 0 & B^t \end{pmatrix} = \det A^t \, \det B^t = \det A \, \det B.$$

□

Suppose that A and B are finite dimensional vector spaces over a field F and $f : A \to B$ is a linear transformation. Then we have the short exact sequence

$$0 \longrightarrow \operatorname{Ker} f \xrightarrow{\ \subseteq\ } A \xrightarrow{\ f\ } \operatorname{Im} f \longrightarrow 0 \cdot$$

If f is 1-to-1, then $\operatorname{Ker} f = 0$. Also, if f is trivial, then $\operatorname{Im} f = 0$. Assume that f is neither 1-to-1 nor trivial. Let $k = (k_1, \ldots, k_r)$, $b = (b_1, \ldots, b_s)$, and $a = (a_1, \ldots, a_{r+s})$ be ordered bases for $\operatorname{Ker} f$, $\operatorname{Im} f$, and A, respectively. Consider $\widetilde{b} = (\widetilde{b}_1, \ldots, \widetilde{b}_s)$ such that $f(\widetilde{b}_i) = b_i$ for each $i \in \{1, \ldots, s\}$. Then we have an ordered basis $(k, \widetilde{b}) = (k_1, \ldots, k_r, \widetilde{b}_1 \ldots, \widetilde{b}_s)$ for A, where $\widetilde{b} = (\widetilde{b}_1, \ldots, \widetilde{b}_s)$ is called a lifting of b by f, or simply, a lifting of b. If b and b' are distinct ordered bases for $\operatorname{Im} f$, then $\widetilde{b} \neq \widetilde{b'}$. However, although $b = b'$, $\widetilde{b}$ and $\widetilde{b'}$ need not be the same. To avoid this ambiguity, we write $\widetilde{b}^{(1)}$ for $\widetilde{b}$ and $\widetilde{b}^{(2)}$ for $\widetilde{b'}$ if $b = b'$.

Lemma 2.4. *Let $f : A \to B$ be a linear transformation, and let $k = (k_1, \ldots, k_r)$ be an ordered basis for $\operatorname{Ker} f$, and let $b = (b_1, \ldots, b_s)$ be an ordered basis for $\operatorname{Im} f$. Then if $\widetilde{b}^{(1)}$ and $\widetilde{b}^{(2)}$ are liftings of b, then*

$$(k, \widetilde{b}^{(1)}) \sim (k, \widetilde{b}^{(2)}).$$

In other words, the equivalence class of $(k, \widetilde{b})$ *with respect to* $\sim$ *does not depend on the choice of lifting* $\widetilde{b}$ *of* b*. We denote the equivalence class of* $(k, \widetilde{b})$ *by* kb*.*

Proof. The transition matrix $((k, \widetilde{b}^{(1)})/(k, \widetilde{b}^{(2)}))$ is $\begin{pmatrix} I_r & 0 \\ C & I_s \end{pmatrix}$ for some $C \in M_{s\times r}(F)$. Therefore, $[(k, \widetilde{b}^{(1)})/(k, \widetilde{b}^{(2)})] = 1$. That is, $(k, \widetilde{b}^{(1)}) \sim (k, \widetilde{b}^{(2)})$. □

Lemma 2.5. *Let* $f : A \to B$ *be a linear transformation, and let* $k = (k_1, \dots, k_r)$ *and* $k' = (k'_1, \dots, k'_r)$ *be ordered bases for* $\operatorname{Ker} f$*, and let* $b = (b_1, \dots, b_s)$ *and* $b' = (b'_1, \dots, b'_s)$ *be distinct ordered bases for* $\operatorname{Im} f$*. Then*

(1) if $\widetilde{b}^{(1)}$ *and* $\widetilde{b}^{(2)}$ *are liftings of* b*, then* $[(k, \widetilde{b}^{(1)})/(k', \widetilde{b}^{(2)})] = [k/k']$*;*

(2) if $\widetilde{b}$ *is a lifting of* b *and* $\widetilde{b'}$ *is a lifting of* b'*, then* $[(k, \widetilde{b})/(k, \widetilde{b'})] = [b/b']$*.*

Proof. (1) The transition matrix $((k, \widetilde{b}^{(1)})/(k', \widetilde{b}^{(2)}))$ is $\begin{pmatrix} (k/k') & 0 \\ C & I_s \end{pmatrix}$ for some $C \in M_{s\times r}(F)$. Therefore, $[(k, \widetilde{b}^{(1)})/(k', \widetilde{b}^{(2)})] = [k/k']$. (2) Similarly, the transition matrix $((k, \widetilde{b}^{(1)})/(k, \widetilde{b}^{(2)}))$ is $\begin{pmatrix} I_r & 0 \\ C & (b/b') \end{pmatrix}$ for some $C \in M_{s\times r}(F)$, so $[(k, \widetilde{b})/(k, \widetilde{b'})] = [b/b']$. □

Notation 2.6. *By Lemma 2.2 and Lemma 2.4, we can express the results of Lemma 2.5 in terms of equivalent classes as follows:*

(1) $[kb/k'b] = [(k, \widetilde{b}^{(1)})/(k', \widetilde{b}^{(2)})] = [k/k']$*; (2)* $[kb/kb'] = [(k, \widetilde{b})/(k, \widetilde{b'})] = [b/b']$*.*

Corollary 2.7. *Let* $f : A \to B$ *be a linear transformation. Then if* k *and* k' *are ordered bases for* $\operatorname{Ker} f$ *and* b *and* b' *are distinct ordered bases for* $\operatorname{Im} f$ *and* $\widetilde{b}$ *is a lifting of* b *and* $\widetilde{b'}$ *is a lifting of* b'*, then*

$$[(k, \widetilde{b})/(k', \widetilde{b'})] = [k/k'][b/b'].$$

Proof. By Lemma 2.5, $[(k, \widetilde{b})/(k', \widetilde{b'})] = [(k, \widetilde{b})/(k', \widetilde{b})][(k', \widetilde{b})/(k', \widetilde{b'})] = [k/k'][b/b']$. □

Notation 2.8. *The result of Corollary 2.7 can be written as*

$$[kb/k'b'] = [(k, \widetilde{b})/(k', \widetilde{b'})] = [k/k'][b/b'].$$

Lemma 2.9. *Let* A *and* B *be finite dimensional vector spaces over a field* F*. Then if* a *and* a' *are ordered bases for* A *and* b *and* b' *are ordered bases for* B*, then* (a, b) *and* (a', b') *are ordered bases for* $A \oplus B$ *and* $[(a, b)/(a', b')] = [a/a'][b/b']$*.*

Proof. The transition matrix from (a, b) to (a', b') is $((a, b)/(a', b')) = \begin{pmatrix} (a/a') & 0 \\ 0 & (b/b') \end{pmatrix}$. Therefore, $[(a, b)/(a', b')] = [a/a'][b/b']$. □

We introduce the required definitions and properties from algebraic topology and homological algebra. See [1].

Definition 2.10. Let $C_0, \dots, C_m$ be modules over a ring R, and let $\partial_i : C_{i+1} \to C_i$ be a R-module homomorphism for each $i \in \{0, \dots, m-1\}$. Then

$$C = \left(0 \longrightarrow C_m \xrightarrow{\partial_{m-1}} C_{m-1} \xrightarrow{\partial_{m-2}} \cdots\cdots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \longrightarrow 0 \right)$$

is called a chain complex of length m over R if $\partial_{i-1} \circ \partial_i = 0$ for each $i \in \{0, \dots, m\}$. Note that $C_{-1} = C_{m+1} = 0$ and $\partial_{-1} = \partial_m = 0$. Also, we write the R-modules $Z_i(C) = \operatorname{Ker} \partial_{i-1}$, $B_i(C) = \operatorname{Im} \partial_i$, and $H_i(C) = Z_i(C)/B_i(C)$ for each $i \in \{0, \dots, m\}$ which are called the i-th cycle, the i-th boundary, and the i-th homology of the chain complex C, respectively. In particular, a chain complex C is said to be acyclic if $H_i(C) = 0$ for each i.

Note that $\partial_{i-1} \circ \partial_i = 0$ if and only if $\operatorname{Im} \partial_i \subseteq \operatorname{Ker} \partial_{i-1}$ and $H_i(C) = 0$ if and only if $\operatorname{Im} \partial_i = \operatorname{Ker} \partial_{i-1}$ for each $i \in \{0, \dots, m\}$.

Definition 2.11. A chain complex C over a field F is said to be based if C_i has a distinguished basis c_i for each i.

Remark that we can think of a chain complex C of length m as the chain complex C of length n for any $n \geq m$ by letting $C_{m+1} = \cdots = C_n = 0$ and $\partial_m = \cdots = \partial_{n-1} = 0$. For this reason, when we consider finitely many chain complexes with different lengths simultaneously, we assume that they have the same length m which is the greatest length of them.

Definition 2.12. Let C and C' be chain complexes of length m over a ring R. Then a sequence $f = (f_i : C_i \to C'_i)_{i=0}^m$ of R-module homomorphisms is called a chain map from C to C', denoted by $f : C \to C'$, if $\partial'_{i-1} \circ f_i = f_{i-1} \circ \partial_{i-1}$ for each $i \in \{1, \dots, m\}$. Note that $f_{-1} = 0$ and $f_{m+1} = 0$.

$$\begin{array}{ccccccccc}
\cdots\cdots & \xrightarrow{\partial_{i+1}} & C_{i+1} & \xrightarrow{\partial_i} & C_i & \xrightarrow{\partial_{i-1}} & C_{i-1} & \xrightarrow{\partial_{i-2}} & \cdots\cdots \\
\big\downarrow & & \big\downarrow{\scriptstyle f_{i+1}} & & \big\downarrow{\scriptstyle f_i} & & \big\downarrow{\scriptstyle f_{i-1}} & & \big\downarrow \\
\cdots\cdots & \xrightarrow{\partial'_{i+1}} & C'_{i+1} & \xrightarrow{\partial'_i} & C'_i & \xrightarrow{\partial'_{i-1}} & C'_{i-1} & \xrightarrow{\partial'_{i-2}} & \cdots\cdots
\end{array}$$

Proposition 2.13. *Let C and C' be chain complexes of length m over a ring R. Then a chain map $f : C \to C'$ induces a unique sequence $f_* = (f_{i*} : H_i(C) \to H_i(C'))_{i=0}^m$ of homomorphisms, denoted by $f_* : H_*(C) \to H_*(C')$, such that for each $i \in \{0, \dots, m\}$, $f_{i*} : H_i(C) \to H_i(C')$ is defined by $f_{i*}([z]) = [f_i(z)]$ for all $z \in Z_i(C)$. We call $f_* : H_*(C) \to H_*(C')$ the induced homomorphism of f.*

Proof. It suffices to show that for each $i \in \{0, \dots, m\}$, $f_i(z) \in Z_i(C')$ for all $z \in Z_i(C)$ and $f_i(b) \in B_i(C')$ for all $b \in B_i(C)$. Let $z \in Z_i(C)$. Then $(f_{i-1} \circ \partial_{i-1})(z) = f_{i-1}(0) = 0$, hence, $(\partial'_{i-1} \circ f_i)(z) = 0$. That is, $f_i(z) \in Z_i(C')$. Let $b \in B_i(C)$. Then $b = \partial_i(c)$ for some $c \in C_{i+1}$, hence, $f_i(b) = (f_i \circ \partial_i)(c) = (\partial'_i \circ f_{i+1})(c) = \partial'_i(f_{i+1}(c))$. Since $f_{i+1}(c) \in C'_{i+1}$, $f_i(b) \in B_i(C')$. Therefore, we have a unique homomorphism $f_{i*} : H_i(C) \to H_i(C')$. □

Definition 2.14. Let C and C' be chain complexes of length m over a ring R, and let $f : C \to C'$ and $g : C \to C'$ be chain maps. Then f and g are said to be chain homotopic, denoted by $f \simeq g$, if there is a sequence $T = (T_i : C_i \to C'_{i+1})_{i=-1}^m$ of homomorphisms such that $f_i - g_i = \partial'_i \circ T_i + T_{i-1} \circ \partial_{i-1}$ for each $i \in \{0, \dots, m\}$. Note that $T_{-1} = 0$. When such a T is known, which is called a chain homotopy between f and g, we say that f and g are chain homotopic by T.

Proposition 2.15. *If C and C' are chain complexes of length m over a ring R, then the chain homotopic relation $\simeq$ on the set $[C, C']$ of all chain maps from C to C' is an equivalence relation.*

Proof. For each $f \in [C, C']$, $f \simeq f$ by zero map. That is, $T = 0$. If $f \simeq g$ in $[C, C']$ by T, then $g \simeq f$ by $-T$. If $f \simeq g$ in $[C, C']$ by T_1 and $g \simeq h$ in $[C, C']$ by T_2, then $f \simeq h$ by $T_1 + T_2$. Hence, $\simeq$ is an equivalence relation on $[C, C']$. □

Proposition 2.16. *Let C, C', and C'' be chain complexes of length m over a ring R, and let $f : C \to C'$, $g : C \to C'$, $f' : C' \to C''$, and $g' : C' \to C''$ be chain maps. Then if $f \simeq g$ and $f' \simeq g'$, then $f' \circ f \simeq g' \circ g : C \to C''$.*

Proof. Suppose that $f \simeq g$ and $T = (T_i : C_i \to C'_{i+1})_{i=-1}^m$ is a sequence of homomorphisms such that $f_i - g_i = \partial'_i \circ T_i + T_{i-1} \circ \partial_{i-1}$ for each $i \in \{0, \dots, m\}$. Let $i \in \{0, \dots, m\}$ and $c \in C_i$. Then $f_i(c) - g_i(c) = \partial'_i(T_i(c)) + T_{i-1}(\partial_{i-1}(c))$. Hence,

$$\begin{aligned}(f'_i \circ f_i)(c) - (f'_i \circ g_i)(c) &= ((f'_i \circ \partial'_i) \circ T_i)(c) + ((f'_i \circ T_{i-1}) \circ \partial_{i-1})(c) \\ &= ((\partial''_i \circ f'_{i+1}) \circ T_i)(c) + ((f'_i \circ T_{i-1}) \circ \partial_{i-1})(c) \\ &= (\partial''_i \circ (f'_{i+1} \circ T_i))(c) + ((f'_i \circ T_{i-1}) \circ \partial_{i-1})(c).\end{aligned}$$

Therefore, $f' \circ f \simeq f' \circ g : C \to C''$ by $f' \circ T = (f'_{i+1} \circ T_i : C_i \to C''_{i+1})_{i=-1}^m$.

Similarly, we show that $f' \circ g \simeq g' \circ g : C \to C''$.

Suppose that $f' \simeq g'$ and $T' = (T'_i : C'_i \to C''_{i+1})_{i=-1}^m$ is a sequence of homomorphisms such that $f'_i - g'_i = \partial''_i \circ T'_i + T'_{i-1} \circ \partial'_{i-1}$ for each $i \in \{0, \dots, m\}$. Let $i \in \{0, \dots, m\}$ and $c \in C_i$. Hence,

$$\begin{aligned}(f'_i \circ g_i)(c) - (g'_i \circ g_i)(c) &= (\partial''_i \circ T'_i)(g_i(c)) + (T'_{i-1} \circ \partial'_{i-1})(g_i(c)) \\ &= (\partial''_i \circ (T'_i \circ g_i))(c) + (T'_{i-1} \circ (\partial'_{i-1} \circ g_i))(c) \\ &= (\partial''_i \circ (T'_i \circ g_i))(c) + (T'_{i-1} \circ (g_{i-1} \circ \partial_{i-1}))(c) \\ &= (\partial''_i \circ (T'_i \circ g_i))(c) + ((T'_{i-1} \circ g_{i-1}) \circ \partial_{i-1})(c).\end{aligned}$$

Therefore, $f' \circ g \simeq g' \circ g : C \to C''$ by $T' \circ g = (T'_i \circ g_i : C_i \to C''_{i+1})_{i=-1}^m$. Furthermore, $f' \circ f \simeq g' \circ g : C \to C''$ by $f' \circ T + T' \circ g$. □

Proposition 2.17. *Let C and C' be chain complexes of length m over a field F. Then if chain maps $f : C \to C'$ and $g : C \to C'$ are chain homotopic, then the induced homomorphisms $f_* = g_* : H_*(C) \to H_*(C')$.*

Proof. Suppose that f and g are chain homotopic and $T = (T_i : C_i \to C'_{i+1})_{i=-1}^m$ is a sequence of homomorphisms such that $f_i - g_i = \partial'_i \circ T_i + T_{i-1} \circ \partial_{i-1}$ for each $i \in \{0, \dots, m\}$. Let $i \in \{0, \dots, m\}$ and $z \in Z_i(C)$. Then $f_i(z) - g_i(z) = \partial'_i(T_i(z)) + T_{i-1}(\partial_{i-1}(z)) = \partial'_i(T_i(z)) + T_{i-1}(0) = \partial'_i(T_i(z)) \in B_i(C')$. Hence, $f_{i*}([z]) = [f_i(z)] = [g_i(z)] = g_{i*}([z])$. Therefore, $f_* = g_*$. □

Definition 2.18. Let C and C' be chain complexes of length m over a field F. Then C and C' are said to be chain equivalent if there are chain maps $f : C \to C'$ and $g : C' \to C$ such that $g \circ f \simeq I_C$ and $f \circ g \simeq I_{C'}$, where $I_C : C \to C$ and $I_{C'} : C' \to C'$ are the identity chain maps. When such f and g are known, which are called the chain equivalences between C and C', we say that C and C' are chain equivalent by (f, g).

Proposition 2.19. *The chain equivalent relation $\simeq$ on the set K^m of all chain complexes of length m over a ring R is an equivalence relation.*

Proof. For each $C \in K^m$, $C \simeq C$ by (I_C, I_C). If $C \simeq C'$ in K^m by (f, g), then $C' \simeq C$ by (g, f). If $C \simeq C'$ in K^m by (f, g) and $C' \simeq C''$ in K^m by (h, k), then $C \simeq C''$ by $(h \circ f, k \circ g)$ by Proposition 2.16. □

3. The torsion of a quasi-isomorphism

In this section, we define a quasi-isomorphism and the torsion of it.

Definition 3.1. Let C and C' be chain complexes of length m over a ring R. Then a chain map $f : C \to C'$ is said to be a quasi-isomorphism if the induced homomorphism $f_* : H_*(C) \to H_*(C')$ between homology is an isomorphism. That is, the induced homomorphism $f_{i*} : H_i(C) \to H_i(C')$ is an isomorphism for each $i \in \{0, \dots, m\}$.

Proposition 3.2. *Let C and C' be chain complexes of length m over a ring R. Then if a chain map $f : C \to C'$ is an isomorphism, then $f : C \to C'$ is a quasi-isomorphism.*

Proof. We show that the induced homomorphism $f_{i*} : H_i(C) \to H_i(C')$ is an isomorphism for each i. Suppose that $z \in f_i^{-1}(B_i(C')) \cap Z_i(C)$. Then $f_i(z) \in B_i(C')$ and $\partial_{i-1}(z) = 0$. Let $w \in D_{i+1}$ such that $f_i(z) = \partial'_i(w)$. Since f_{i+1} is onto, we can choose $x \in C_{i+1}$ so that $w = f_{i+1}(x)$. Hence, $\partial'_i(f_{i+1}(x)) = f_i(z) = f_i(\partial_i(x))$. Since f_i is 1-to-1, $z = \partial_i(x) \in B_i(C)$. Hence, f_{i*} is 1-to-1.

Suppose that $z' \in Z_i(C')$. Since f_i is onto, we can choose $z \in C_i$ so that $z' = f_i(z)$. Then $\partial'_{i-1}(f_i(z)) = 0 = f_{i-1}(\partial_{i-1}(z))$. Since f_{i-1} is 1-to-1, $\partial_{i-1}(z) = 0$, that is, $z \in Z_i(C)$. Hence $f_i : Z_i(C) \to Z_i(C')$ is onto. Therefore, f_{i*} is onto. $\square$

To define the torsion of a quasi-isomorphism $f : C \to C'$, we use the following short exact sequences

$$0 \longrightarrow Z_i(C) \xrightarrow{\subseteq} C_i \xrightarrow{\partial_{i-1}} B_{i-1}(C) \longrightarrow 0,$$

$$0 \longrightarrow B_i(C) \xrightarrow{\subseteq} Z_i(C) \xrightarrow{\pi} H_i(C) \longrightarrow 0,$$

$$0 \longrightarrow Z_i(C') \xrightarrow{\subseteq} C'_i \xrightarrow{\partial'_{i-1}} B_{i-1}(C') \longrightarrow 0,$$

$$0 \longrightarrow B_i(C') \xrightarrow{\subseteq} Z_i(C') \xrightarrow{\pi} H_i(C') \longrightarrow 0$$

for each i, where π is the canonical map.

Definition 3.3. Let C and C' be based chain complexes of length m over a field F such that C_i and C'_i are finite dimensional vector spaces for each $i \in \{0, \ldots, m\}$, and let $f : C \to C'$ be a quasi-isomorphism. Then the torsion $\tau(f)$ of f is defined by

$$\tau(f) = \prod_{i=0}^{m} \left(\frac{[(b_i h_i) b_{i-1}/c_i]}{[(b'_i f_{i*}(h_i)) b'_{i-1}/c'_i]} \right)^{(-1)^{i+1}},$$

where b_i, b_{i-1}, b'_i, b'_{i-1}, c_i, c'_i, and h_i are bases for $B_i(C)$, $B_{i-1}(C)$, $B_i(C')$, $B_{i-1}(C')$, C_i, C'_i, and $H_i(C)$, respectively, for each $i \in \{0, \ldots, m\}$. Note that $b_{-1} = b_m = b'_{-1} = b'_m = \emptyset$.

We can also write the torsion $\tau(f)$ of f by

$$\tau(f) = \prod_{i=0}^{m} \left(\frac{[(b_i,\, \widetilde{h_i},\, \widetilde{b_{i-1}})/c_i]}{[(b'_i,\, f_i(\widetilde{h_i}),\, \widetilde{b'_{i-1}})/c'_i]} \right)^{(-1)^{i+1}},$$

where b_i, b_{i-1}, b'_i, b'_{i-1}, c_i, c'_i, and h_i are bases for $B_i(C)$, $B_{i-1}(C)$, $B_i(C')$, $B_{i-1}(C')$, C_i, C'_i, and $H_i(C)$, respectively, and $\widetilde{b_{i-1}}$, $\widetilde{b'_{i-1}}$, $\widetilde{h_i}$, and $f_i(\widetilde{h_i})$ are liftings of b_{i-1}, b'_{i-1}, h_i, and $f_{i*}(h_i)$, respectively, for each $i \in \{0, \ldots, m\}$.

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & C_m & \xrightarrow{\partial_{m-1}} & C_{m-1} & \xrightarrow{\partial_{m-2}} & \cdots\cdots & \xrightarrow{\partial_1} & C_1 & \xrightarrow{\partial_0} & C_0 & \longrightarrow & 0 \\
\downarrow & & {\scriptstyle f_m}\downarrow & & {\scriptstyle f_{m-1}}\downarrow & & \downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & \downarrow \\
0 & \longrightarrow & C'_m & \xrightarrow{\partial'_{m-1}} & C'_{m-1} & \xrightarrow{\partial'_{m-2}} & \cdots\cdots & \xrightarrow{\partial'_1} & C'_1 & \xrightarrow{\partial'_0} & C'_0 & \longrightarrow & 0
\end{array}$$

Lemma 3.4. *$\tau(f)$ dose not depend on the choices of b_i, b'_i, and h_i. That is, the torsion τ on quasi-isomorphisms is well-defined.*

Proof. We use Notation 2.6 and 2.8 to prove this lemma.

For each $i \in \{0, \dots, m\}$, let b_i, h_i, and b'_i be bases for $B_i(C)$, $H_i(C)$, and $B_i(C')$, respectively. Then

$$\tau(f) = \prod_{i=0}^{m} \left(\frac{[(b_i h_i) b_{i-1}/c_i]}{[(b'_i f_{i*}(h_i)) b'_{i-1}/c'_i]} \right)^{(-1)^{i+1}}.$$

Step 1. Show that $\tau(f)$ is independent of the choice of h_i.

Let $i \in \{0, \dots, m\}$, and let h'_i be a basis for $H_i(C)$. Then we have

$[(b_i h'_i) b_{i-1}/c_i] = [(b_i h'_i) b_{i-1}/(b_i h_i) b_{i-1}][(b_i h_i) b_{i-1}/c_i] = [b_i h'_i/b_i h_i][(b_i h_i) b_{i-1}/c_i]$
$= [h'_i/h_i][(b_i h_i) b_{i-1}/c_i]$ and
$[(b'_i f_{i*}(h'_i)) b'_{i-1}/c'_i] = [(b'_i f_{i*}(h'_i)) b'_{i-1}/(b'_i f_{i*}(h_i)) b'_{i-1}][(b'_i f_{i*}(h_i)) b'_{i-1}/c'_i]$
$= [b'_i f_{i*}(h'_i)/b'_i f_{i*}(h_i)][(b'_i f_{i*}(h_i)) b'_{i-1}/c'_i]$
$= [f_{i*}(h'_i)/f_{i*}(h_i)][(b'_i f_{i*}(h_i)) b'_{i-1}/c'_i]$.

Since f_{i*} is an isomorphism, $(h'_i/h_i) = (f_{i*}(h'_i)/f_{i*}(h_i))$, so $[h'_i/h_i] = [f_{i*}(h'_i)/f_{i*}(h_i)]$. Hence, we have

$$\frac{[(b_i h'_i) b_{i-1}/c_i]}{[(b'_i f_{i*}(h'_i)) b'_{i-1}/c'_i]} = \frac{[(b_i h_i) b_{i-1}/c_i]}{[(b'_i f_{i*}(h_i)) b'_{i-1}/c'_i]}$$

for each $i \in \{0, \dots, m\}$.

Therefore, $\tau(f)$ does not depend on the choice of bases for homology spaces.

Step 2. Show that $\tau(f)$ is independent of the choices of b_i and b'_i.

For each $i \in \{0, \dots, m-1\}$, let $X_i(b_{i-1}, b'_{i-1}, b_i, b'_i, b_{i+1}, b'_{i+1})$ be the expression

$$\left(\frac{[(b_i h_i) b_{i-1}/c_i]}{[(b'_i f_{i*}(h_i)) b'_{i-1}/c'_i]} \right)^{(-1)^{i+1}} \left(\frac{[(b_{i+1} h_{i+1}) b_i/c_{i+1}]}{[(b'_{i+1} f_{i+1*}(h_{i+1})) b'_i/c'_{i+1}]} \right)^{(-1)^{i+2}}.$$

We claim that $X_i(b_{i-1}, b'_{i-1}, b_i, b'_i, b_{i+1}, b'_{i+1})$ is independent of the choices of b_i and b'_i. Let $i \in \{0, \dots, m-1\}$, and let v_i and v'_i be bases for $B_i(C)$ and $B_i(C')$, respectively. Then we have the following equations

$[(v_i h_i) b_{i-1}/c_i] = [(v_i h_i) b_{i-1}/(b_i h_i) b_{i-1}][(b_i h_i) b_{i-1}/c_i] = [v_i h_i/b_i h_i][(b_i h_i) b_{i-1}/c_i]$
$= [v_i/b_i][(b_i h_i) b_{i-1}/c_i]$,

$[(b_{i+1} h_{i+1}) v_i/c_{i+1}] = [(b_{i+1} h_{i+1}) v_i/(b_{i+1} h_{i+1}) b_i][(b_{i+1} h_{i+1}) b_i/c_{i+1}]$
$= [b_{i+1} h_{i+1}/b_{i+1} h_{i+1}][v_i/b_i][(b_{i+1} h_{i+1}) b_i/c_{i+1}]$
$= [v_i/b_i][(b_{i+1} h_{i+1}) b_i/c_{i+1}]$,

$$
\begin{aligned}
&[(v'_i f_{i*}(h_i))b'_{i-1}/c'_i] = [(v'_i f_{i*}(h_i))b'_{i-1}/(b'_i f_{i*}(h_i))b'_{i-1}][(b'_i f_{i*}(h_i))b'_{i-1}/c'_i]\\
&= [v'_i f_{i*}(h_i)/b'_i f_{i*}(h_i)][(b'_i f_{i*}(h_i))b'_{i-1}/c'_i]\\
&= [v'_i/b'_i][(b'_i f_{i*}(h_i))b'_{i-1}/c'_i],
\end{aligned}
$$

$$
\begin{aligned}
&[(b'_{i+1} f_{i+1*}(h_{i+1}))v'_i/c'_{i+1}]\\
&= [(b'_{i+1} f_{i+1*}(h_{i+1}))v'_i/(b'_{i+1} f_{i+1*}(h_{i+1}))b'_i][(b'_{i+1} f_{i+1*}(h_{i+1}))b'_i/c'_{i+1}]\\
&= [b'_{i+1} f_{i+1*}(h_{i+1})/b'_{i+1} f_{i+1*}(h_{i+1})][v'_i/b'_i][(b'_{i+1} f_{i+1*}(h_{i+1}))b'_i/c'_{i+1}]\\
&= [v'_i/b'_i][(b'_{i+1} f_{i+1*}(h_{i+1}))b'_i/c'_{i+1}].
\end{aligned}
$$

Hence, we have

$$X_i(b_{i-1}, b'_{i-1}, v_i, v'_i, b_{i+1}, b'_{i+1}) = X_i(b_{i-1}, b'_{i-1}, b_i, b'_i, b_{i+1}, b'_{i+1})$$

for each $i \in \{0, \dots, m-1\}$. Also, we have

$$X_i(v_{i-1}, v'_{i-1}, v_i, v'_i, b_{i+1}, b'_{i+1}) = X_i(v_{i-1}, v'_{i-1}, b_i, b'_i, b_{i+1}, b'_{i+1})$$

for each $i \in \{0, \dots, m-1\}$. Remark that $v_{-1} = v_m = v'_{-1} = v'_m = \emptyset$. Therefore, by these facts, we conclude that

$$\prod_{i=0}^{m} \left(\frac{[(v_i h_i)v_{i-1}/c_i]}{[(v'_i f_{i*}(h_i))v'_{i-1}/c'_i]} \right)^{(-1)^{i+1}} = \prod_{i=0}^{m} \left(\frac{[(b_i h_i)b_{i-1}/c_i]}{[(b'_i f_{i*}(h_i))b'_{i-1}/c'_i]} \right)^{(-1)^{i+1}}.$$

This proves the lemma. □

Lemma 3.5. *Let C, C', and C'' be based chain complexes of length m over a field F. Then if $f : C \to C'$ and $g : C' \to C''$ are quasi-isomorphisms, then $g \circ f : C \to C''$ is a quasi-isomorphism and $\tau(g \circ f) = \tau(g)\tau(f)$.*

Proof.

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & C_m & \xrightarrow{\partial_{m-1}} & C_{m-1} & \xrightarrow{\partial_{m-2}} & \cdots\cdots & \xrightarrow{\partial_1} & C_1 & \xrightarrow{\partial_0} & C_0 & \longrightarrow & 0\\
\big\downarrow & & {\scriptstyle f_m}\big\downarrow & & {\scriptstyle f_{m-1}}\big\downarrow & & \big\downarrow & & {\scriptstyle f_1}\big\downarrow & & {\scriptstyle f_0}\big\downarrow & & \big\downarrow\\
0 & \longrightarrow & C'_m & \xrightarrow{\partial'_{m-1}} & C'_{m-1} & \xrightarrow{\partial'_{m-2}} & \cdots\cdots & \xrightarrow{\partial'_1} & C'_1 & \xrightarrow{\partial'_0} & C'_0 & \longrightarrow & 0\\
\big\downarrow & & {\scriptstyle g_m}\big\downarrow & & {\scriptstyle g_{m-1}}\big\downarrow & & \big\downarrow & & {\scriptstyle g_1}\big\downarrow & & {\scriptstyle g_0}\big\downarrow & & \big\downarrow\\
0 & \longrightarrow & C''_m & \xrightarrow{\partial''_{m-1}} & C''_{m-1} & \xrightarrow{\partial''_{m-2}} & \cdots\cdots & \xrightarrow{\partial''_1} & C''_1 & \xrightarrow{\partial''_0} & C''_0 & \longrightarrow & 0
\end{array}
$$

For each $i \in \{0, \dots, m\}$, $\partial''_{i-1} \circ (g_i \circ f_i) = (g_{i-1} \circ f_{i-1}) \circ \partial_{i-1}$ and $(g_i \circ f_i)_* = g_{i*} \circ f_{i*} : H_i(C) \to H_i(C'')$ is an isomorphism. Hence, $g \circ f : C \to C''$ is a quasi-isomorphism. Note that the torsion does not depend on the choice of basis for homology.

Suppose that

$$\tau(f)=\prod_{i=0}^{m}\left(\frac{[(b_ih_i)b_{i-1}/c_i]}{[(b'_if_{i*}(h_i))b'_{i-1}/c'_i]}\right)^{(-1)^{i+1}}.$$

Then

$$\tau(g)=\prod_{i=0}^{m}\left(\frac{[(b'_if_{i*}(h_i))b'_{i-1}/c'_i]}{[(b''_ig_{i*}(f_{i*}(h_i)))b''_{i-1}/c''_i]}\right)^{(-1)^{i+1}}.$$

Hence,

$$\begin{aligned}\tau(g)\tau(f)=\tau(f)\tau(g)&=\prod_{i=0}^{m}\left(\frac{[(b_ih_i)b_{i-1}/c_i]}{[(b''_ig_{i*}(f_{i*}(h_i)))b''_{i-1}/c''_i]}\right)^{(-1)^{i+1}}\\&=\prod_{i=0}^{m}\left(\frac{[(b_ih_i)b_{i-1}/c_i]}{[(b''_i(g_i\circ f_i)_*(h_i))b''_{i-1}/c''_i]}\right)^{(-1)^{i+1}}=\tau(g\circ f).\end{aligned}$$

□

Lemma 3.6. *Let C, C', C'', and C''' be based chain complexes of length m over a field F. Then if $f:C\to C'$ and $g:C''\to C'''$ are quasi-isomorphisms, then $f\oplus g:C\oplus C''\to C'\oplus C'''$ is a quasi-isomorphism and $\tau(f\oplus g)=\pm\,\tau(f)\tau(g)$.*

Proof. For each $i\in\{0,\dots,m\}$, $(\partial'_{i-1}\oplus\partial'''_{i-1})\circ(f_i\oplus g_i)=(\partial'_{i-1}\circ f_i)\oplus(\partial'''_{i-1}\circ g_i)=(f_{i-1}\circ\partial_{i-1})\oplus(g_{i-1}\circ\partial''_{i-1})=(f_{i-1}\oplus g_{i-1})\circ(\partial_{i-1}\oplus\partial''_{i-1})$ and $(f_i\oplus g_i)_*:H_i(C\oplus C'')\to H_i(C'\oplus C''')$ is an isomorphism since $f_{i*}\oplus g_{i*}:H_i(C)\oplus H_i(C'')\to H_i(C')\oplus H_i(C''')$ is an isomorphism. Hence, $f\oplus g:C\oplus C''\to C'\oplus C'''$ is a quasi-isomorphism. Also,

$$\begin{aligned}\tau(f\oplus g)&=\prod_{i=0}^{m}\left(\frac{[((b_i\oplus b''_i)(h_i\oplus h''_i))(b_{i-1}\oplus b''_{i-1})/(c_i\oplus c''_i)]}{[((b'_i\oplus b'''_i)(f_i\oplus g_i)_*(h_i\oplus h''_i))(b'_{i-1}\oplus b'''_{i-1})/(c'_i\oplus c'''_i)]}\right)^{(-1)^{i+1}}\\&=\prod_{i=0}^{m}\left(\frac{[((b_i\oplus b''_i)(h_i\oplus h''_i))(b_{i-1}\oplus b''_{i-1})/(c_i\oplus c''_i)]}{[((b'_i\oplus b'''_i)(f_{i*}(h_i)\oplus g_{i*}(h''))(b'_{i-1}\oplus b'''_{i-1})/(c'_i\oplus c'''_i)]}\right)^{(-1)^{i+1}}\\&=\pm\prod_{i=0}^{m}\left(\frac{[((b_ih_i)b_{i-1}\oplus(b''_ih''_i)b''_{i-1})/(c_i\oplus c''_i)]}{[(b'_if_{i*}(h_i))b'_{i-1}\oplus(b'''_ig_{i*}(h''_i))b'''_{i-1}/(c'_i\oplus c'''_i)]}\right)^{(-1)^{i+1}}\\&=\pm\prod_{i=0}^{m}\left(\frac{[(b_ih_i)b_{i-1}/c_i]}{[(b'_if_{i*}(h_i))b'_{i-1}/c'_i]}\frac{[(b''_ih''_i)b''_{i-1}/c''_i]}{[(b'''_ig_{i*}(h''_i))b'''_{i-1}/c'''_i]}\right)^{(-1)^{i+1}}\\&=\pm\prod_{i=0}^{m}\left(\frac{[(b_ih_i)b_{i-1}/c_i]}{[(b'_if_{i*}(h_i))b'_{i-1}/c'_i]}\right)^{(-1)^{i+1}}\prod_{i=0}^{m}\left(\frac{[(b''_ih''_i)b''_{i-1}/c''_i]}{[(b'''_ig_{i*}(h''_i))b'''_{i-1}/c'''_i]}\right)^{(-1)^{i+1}}\\&=\pm\,\tau(f)\tau(g).\end{aligned}$$

□

Notice that a sign problem occurs at the 3rd equation. The dimensions of C_i and C'_i are not the same, even boundaries $B_i(C)$ and $B_i(C')$, in general.

To get the exact sign for the torsion of it, for each $i \in \{0, \dots, m\}$, let

$$x_i = \dim_F B_i(C),\ \ x'_i = \dim_F B_i(C'),\ \ x''_i = \dim_F B_i(C''),\ \ x'''_i = \dim_F B_i(C'''),$$
$$y_i = \dim_F H_i(C),\ \ y''_i = \dim_F H_i(C''),$$

Then

$$\tau(f \oplus g) = \frac{(-1)^{\sum_{i=0}^{m}(x''_i y_i + x_{i-1}(x''_i + y''_i))}}{(-1)^{\sum_{i=0}^{m}(x'''_i y_i + x'_{i-1}(x'''_i + y''_i))}}\,\tau(f)\tau(g).$$

Therefore,

$$\tau(f \oplus g) = (-1)^{\sum_{i=0}^{m}[(x''_i y_i + x_{i-1}(x''_i + y''_i)) - (x'''_i y_i + x'_{i-1}(x'''_i + y''_i))]}\,\tau(f)\tau(g).$$

In particular, if $C = C'$ and $C'' = C'''$, then $\tau(f \oplus g) = \tau(f)\tau(g)$.

Corollary 3.7. *Let C, C', C'', and C''' be based chain complexes of length m over a field F. Then if $f : C \to C'$ and $g : C'' \to C'''$ are quasi-isomorphisms, then $\tau(g \oplus f) = \pm\,\tau(f \oplus g)$.*

Proof. Since $\tau(f \oplus g) = \pm\,\tau(f)\tau(g)$ and $\tau(g \oplus f) = \pm\,\tau(g)\tau(f)$, we have $\tau(g \oplus f) = \pm\,\tau(f \oplus g)$. □

Corollary 3.8. *Let C, C', C'', C''', C'''', and C''''' be based chain complexes of length m over a field F. Then if $f : C \to C'$, $g : C'' \to C'''$, and $h : C'''' \to C'''''$ are quasi-isomorphisms, then $\tau((f \oplus g) \oplus h) = \pm\,\tau(f \oplus (g \oplus h))$.*

Proof. Since $\tau((f \oplus g) \oplus h) = \pm\,\tau(f \oplus g)\tau(h) = \pm\,\tau(f)\tau(g)\tau(h)$ and $\tau(f \oplus (g \oplus h)) = \pm\,\tau(f)\tau(g \oplus h) = \pm\,\tau(f)\tau(g)\tau(h)$, we have $\tau((f \oplus g) \oplus h) = \pm\,\tau(f \oplus (g \oplus h))$. □

Lemma 3.9. *Let C and C' be based chain complexes of length m over a field F. Then if quasi-isomorphisms $f : C \to C'$ and $g : C \to C'$ are chain homotopic, then $\tau(f) = \tau(g)$.*

Proof. Since f and g are chain homotopic, $f_* = g_*$. Therefore, $\tau(f) = \tau(g)$. □

Lemma 3.10. *Let C and C' be based chain complexes of length m over a field F. Then if $f : C \to C'$ and $g : C' \to C$ are the chain equivalences between C and C', then f and g are quasi-isomorphisms and $\tau(f) = \tau(g)^{-1}$.*

Proof. Since $g \circ f \simeq I_C$ and $f \circ g \simeq I_{C'}$, we have $g_* \circ f_* = (g \circ f)_* = I_{C*}$ and $f_* \circ g_* = (f \circ g)_* = I_{C'*}$. Note that I_{C*} and $I_{C'*}$ are the identity induced maps. Hence, f_* and g_* are isomorphisms, so f and g are quasi-isomorphisms. Therefore, $\tau(g)\tau(f) = \tau(g \circ f) = \tau(I_C) = 1$. Hence, we have $\tau(f) = \tau(g)^{-1}$. □

Definition 3.11. Let C be a based chain complex of length m over a field F. Then chain maps $f : C \to C$ and $g : C \to C$ are said to be conjugate if there is a chain isomorphism $h : C \to C$ such that $f = h^{-1} \circ g \circ h$.

Lemma 3.12. *Let C be a based chain complex of length m over a field F. Then if quasi-isomorphisms $f : C \to C$ and $g : C \to C$ are conjugate, then $\tau(f) = \tau(g)$.*

Proof. Suppose that $f = h^{-1} \circ g \circ h$ for some chain isomorphism $h : C \to C$. Then $\tau(f) = \tau(h^{-1} \circ g \circ h) = \tau(h^{-1})\tau(g)\tau(h)$. Since h is a chain isomorphism, h and h^{-1} are chain equivalences. Hence, by Lemma 3.10, we have $\tau(f) = \tau(g)$. □

Let us introduce the definition of torsion of a based acyclic chain complex which gives us a motivation to define our torsion of a quasi-isomorphism. See [2]. The torsion of a based acyclic chain complex can be regarded as the torsion of a quasi-isomorphism.

Definition 3.13. Let C be a based acyclic chain complex of length m over a field F such that C_i is a finite dimensional vector space for each $i \in \{0, \ldots, m\}$. Then the torsion $\tau(C)$ of C is defined by

$$\tau(C) = \prod_{i=0}^{m} [b_i b_{i-1}/c_i]^{(-1)^{i+1}},$$

where b_i and c_i are bases for $B_i(C)$ and C_i, respectively, for each $i \in \{0, \ldots, m\}$. Note that $b_{-1} = b_m = \emptyset$. In particular, the zero chain complex of length m, denoted by 0^m, is acyclic and we define $\tau(0^m) = 1$.

Theorem 3.14. *If C and C' are acyclic based chain complexes of length m over a field F and $f : C \to C'$ is a chain map, then f is a quasi-isomorphism and*

$$\tau(f) = \frac{\tau(C)}{\tau(C')}.$$

Proof. Since C and C' are acyclic, $H_i(C) = H_i(C') = 0$ for each $i \in \{0, \ldots, m\}$. Therefore, f is a quasi-isomorphism and

$$\tau(f) = \prod_{i=0}^{m} \left(\frac{[(b_i h_i) b_{i-1}/c_i]}{[(b'_i f_{i*}(h_i)) b'_{i-1}/c'_i]} \right)^{(-1)^{i+1}} = \prod_{i=0}^{m} \left(\frac{[b_i b_{i-1}/c_i]}{[b'_i b'_{i-1}/c'_i]} \right)^{(-1)^{i+1}} = \frac{\tau(C)}{\tau(C')},$$

where b_i, b_{i-1}, b'_i, b'_{i-1}, c_i, and c'_i are bases for $B_i(C)$, $B_{i-1}(C)$, $B_i(C')$, $B_{i-1}(C')$, C_i, and C'_i, respectively, and $h_i = \emptyset$ for each $i \in \{0, \ldots, m\}$. □

Notation 3.15. *Let C be a based chain complex of length m over a field F. Then we denote zero chain maps which are injective and surjective by*

$$0_{0 \to C} : 0^m \to C \ \text{ and } \ 0_{C \to 0} : C \to 0^m,$$

respectively.

Corollary 3.16. *If C is a based acyclic chain complex of length m over a field F, then $0_{0 \to C} : 0^m \to C$ and $0_{C \to 0} : C \to 0^m$ are quasi-isomorphisms and*

$$\tau(C) = \tau(0_{0 \to C})^{-1} = \tau(0_{C \to 0}).$$

Proof. Since C and 0_m are acyclic, by Theorem 3.14, we have

$$\tau(0_{0\to C}) = \frac{\tau(0^m)}{\tau(C)} = \tau(C)^{-1} \text{ and } \tau(0_{C\to 0}) = \frac{\tau(C)}{\tau(0^m)} = \tau(C).$$

□

Theorem 3.17. *Let C and C' be based chain complexes of length m over a field F. Then if C' is acyclic, then the sequence $i : C \to C \oplus C'$ of injection maps and the sequence $p : C \oplus C' \to C$ of projection maps are quasi-isomorphisms and $\tau(i) = \pm\,\tau(C')^{-1}$ and $\tau(p) = \pm\,\tau(C')$.*

Proof. First, we show that p and i are quasi-isomorphisms. Let $j \in \{0, \ldots, m\}$ and $c \in C_j$ and $c' \in C'_j$. Then $((\partial_{j-1} \oplus \partial'_{j-1}) \circ i_j)(c) = (i_{j-1} \circ \partial_{j-1})(c)$ and $(\partial_{j-1} \circ p_j)(c \oplus c') = (p_{j-1} \circ (\partial_{j-1} \oplus \partial'_{j-1}))(c \oplus c')$. Hence, $i : C \to C \oplus C'$ and $p : C \oplus C' \to C$ are chain maps, which are called the injection chain map and the projection chain map, respectively. We can think of $i : C \to C \oplus C'$ and $p : C \oplus C' \to C$ as $I_C \oplus 0_{0\to C'} : C \oplus 0 \to C \oplus C'$ and $I_C \oplus 0_{C'\to 0} : C \oplus C' \to C \oplus 0$, respectively. By Lemma 3.6, $\tau(i) = \pm\,\tau(C')^{-1}$ and $\tau(p) = \pm\,\tau(C')$. □

We need to distinguish a chain complex with distinct bases. Let us use a pair (C, c) for a based chain complex C with a basis c. Also, a chain map $f : C \to C'$ between chain complexes C and C' with bases c and c', respectively, is denoted by $f : (C, c) \to (C', c')$.

Lemma 3.18. *Let C be a chain complex of length m over a field F, and let $I_C : C \to C$ be the identity chain map. Then if $c^{(1)}$ and $c^{(2)}$ are bases for C, then*

$$\tau(I_C : (C, c^{(1)}) \to (C, c^{(2)})) = \prod_{i=0}^{m} [c_i^{(2)}/c_i^{(1)}]^{(-1)^{i+1}}.$$

Proof. Suppose that b_i, b_{i-1}, and h_i are bases for $B_i(C)$, $B_{i-1}(C)$, and $H_i(C)$, respectively, for each $i \in \{0, \ldots, m\}$. Since the torsion is independent of the choice of bases for boundaries, we have

$$\tau(I_C : (C, c^{(1)}) \to (C, c^{(2)})) = \prod_{i=0}^{m} \left(\frac{[(b_i h_i) b_{i-1}/c_i^{(1)}]}{[(b_i I_{Ci*}(h_i)) b_{i-1}/c_i^{(2)}]} \right)^{(-1)^{i+1}}$$

$$= \prod_{i=0}^{m} \left(\frac{[(b_i h_i) b_{i-1}/c_i^{(1)}]}{[(b_i h_i) b_{i-1}/c_i^{(2)}]} \right)^{(-1)^{i+1}} = \prod_{i=0}^{m} [c_i^{(2)}/c_i^{(1)}]^{(-1)^{i+1}}.$$

□

Theorem 3.19. *Let C and C' be chain complexes of length m over a field F, and let $f : C \to C'$ be a quasi-isomorphism. Then if $c^{(1)}$ and $c^{(2)}$ are bases for C and $c'^{(1)}$*

and $c'^{(2)}$ are bases for C', then

$$\tau(f:(C,c^{(2)})\to(C',c'^{(2)}))=\prod_{i=0}^{m}\left(\frac{[c_i^{(1)}/c_i^{(2)}]}{[c_i'^{(1)}/c_i'^{(2)}]}\right)^{(-1)^{i+1}}\tau(f:(C,c^{(1)})\to(C',c'^{(1)})).$$

Proof. Suppose that $f^{(1)}=f:(C,c^{(1)})\to(C',c'^{(1)})$, $f^{(2)}=f:(C,c^{(2)})\to(C',c'^{(2)})$, $I^{(21)}=I_C:(C,c^{(2)})\to(C,c^{(1)})$, and $I'^{(12)}=I_{C'}:(C',c'^{(1)})\to(C',c'^{(2)})$. Since $f^{(2)}=I'^{(12)}\circ f^{(1)}\circ I^{(21)}$, we have $\tau(f^{(2)})=\tau(I'^{(12)})\tau(f^{(1)})\tau(I^{(21)})$ by Lemma 3.5. Hence, by Lemma 3.18,

$$\tau(f^{(2)})=\prod_{i=0}^{m}[c_i'^{(2)}/c_i'^{(1)}]^{(-1)^{i+1}}\tau(f^{(1)})\prod_{i=0}^{m}[c_i^{(1)}/c_i^{(2)}]^{(-1)^{i+1}}.$$

Therefore,

$$\tau(f^{(2)})=\prod_{i=0}^{m}\left(\frac{[c_i^{(1)}/c_i^{(2)}]}{[c_i'^{(1)}/c_i'^{(2)}]}\right)^{(-1)^{i+1}}\tau(f^{(1)}).$$

□

The torsion of a quasi-isomorphism from a based chain complex to itself can be easily calculated. It turns out that the torsion is determined by the determinants of the induced isomorphisms on homology.

Theorem 3.20. *Let C be a based chain complex of length m over a field F, and let $f:C\to C$ be a quasi-isomorphism. Then*

(1) $\tau(f)$ is independent of the choice of the basis c;

(2) if h_i is a basis for $H_i(C)$ for each $i\in\{0,\dots,m\}$, then

$$\tau(f)=\prod_{i=0}^{m}[h_i/f_{i*}(h_i)]^{(-1)^{i+1}}=\frac{\prod_{i\,even}\det f_{i*}}{\prod_{i\,odd}\det f_{i*}},$$

where $\det f_{i*}=[f_{i*}(h_i)/h_i]$ *for each $i\in\{0,\dots,m\}$.*

Proof. (1) If $c^{(1)}$ and $c^{(2)}$ are bases for C, then, by Theorem 3.19,

$$\tau(f:(C,c^{(2)})\to(C,c^{(2)}))=\prod_{i=0}^{m}\left(\frac{[c_i^{(1)}/c_i^{(2)}]}{[c_i^{(1)}/c_i^{(2)}]}\right)^{(-1)^{i+1}}\tau(f:(C,c^{(1)})\to(C,c^{(1)})).$$

Hence, $\tau(f:(C,c^{(2)})\to(C,c^{(2)}))=\tau(f:(C,c^{(1)})\to(C,c^{(1)}))$.

(2)

$$\tau(f) = \prod_{i=0}^{m} \left(\frac{[(b_i h_i) b_{i-1}/c_i]}{[(b_i f_{i*}(h_i)) b_{i-1}/c_i]} \right)^{(-1)^{i+1}} = \prod_{i=0}^{m} ([(b_i h_i) b_{i-1}/c_i][c_i/(b_i f_{i*}(h_i)) b_{i-1}])^{(-1)^{i+1}}$$
$$= \prod_{i=0}^{m} [(b_i h_i) b_{i-1}/(b_i f_{i*}(h_i)) b_{i-1}]^{(-1)^{i+1}} = \prod_{i=0}^{m} [h_i/f_{i*}(h_i)]^{(-1)^{i+1}}.$$

Also, if $i \in \{1, \ldots, m\}$ is even, then $[h_i/f_{i*}(h_i)]^{(-1)^{i+1}} = [f_{i*}(h_i)/h_i]$. Similarly, if $i \in \{1, \ldots, m\}$ is odd, then $[h_i/f_{i*}(h_i)]^{(-1)^{i+1}} = [h_i/f_{i*}(h_i)] = [f_{i*}(h_i)/h_i]^{-1}$. Notice that $[h_i/f_{i*}(h_i)] = (\det f_{i*})^{-1}$ for each $i \in \{0, \ldots, m\}$. Therefore,

$$\tau(f) = \prod_{i=0}^{m} [h_i/f_{i*}(h_i)]^{(-1)^{i+1}} = \prod_{i=0}^{m} \left(\frac{1}{\det f_{i*}} \right)^{(-1)^{i+1}} = \frac{\prod_{i\, even} \det f_{i*}}{\prod_{i\, odd} \det f_{i*}}.$$

□

The following theorem is a statement which can be regarded as the generalization of Proposition 2.3 to the torsion of quasi-isomorphisms.

Theorem 3.21. *Let C and C' be based chain complexes of length m over a field F, and let $f : C \to C$ and $f' : C' \to C'$ be quasi-isomorphisms, and let $g : C \to C'$ be a chain map. Define $\bar{f} : C \oplus C' \to C \oplus C'$ by $\bar{f}_i(x \oplus y) = f_i(x) \oplus (g_i(x) + f'_i(y))$ for all $x \in C_i$ and $y \in C'_i$ for each $i \in \{0, \ldots, m\}$. Then $\bar{f} : C \oplus C' \to C \oplus C'$ is a quasi-isomorphism and $\tau(\bar{f}) = \tau(f)\tau(f')$. In particular, if $g = 0$, then $\bar{f} = f \oplus f'$.*

Proof. For convenience, let us use $x \oplus y$ for an ordered pair (x, y).

Step 1. Show that $\bar{f} : C \oplus C' \to C \oplus C'$ is a chain map.

Let $i \in \{0, \ldots, m\}$. Suppose that $x, x_1, x_2 \in C$ and $y, y_1, y_2 \in C'$ and $r \in F$. Then we have

$$\bar{f}_i(x_1 \oplus y_1 + x_2 \oplus y_2) = \bar{f}_i((x_1 + x_2) \oplus (y_1 + y_2))$$
$$= f_i(x_1 + x_2) \oplus (g_i(x_1 + x_2) + f'_i(y_1 + y_2))$$
$$= (f_i(x_1) \oplus (g_i(x_1) + f'_i(y_1))) + (f_i(x_2) \oplus (g_i(x_2) + f'_i(y_2)))$$
$$= \bar{f}_i(x_1 \oplus y_1) + \bar{f}_i(x_2 \oplus y_2)$$

and

$$\bar{f}_i(r(x \oplus y)) = \bar{f}_i(rx \oplus ry) = f_i(rx) \oplus (g_i(rx) + f'_i(ry)) = r\bar{f}_i(x \oplus y)$$

and

$$((\partial_{i-1} \oplus \partial'_{i-1}) \circ \bar{f}_i)(x \oplus y) = (\partial_{i-1} \oplus \partial'_{i-1})(f_i(x) \oplus (g_i(x) + f'_i(y)))$$
$$= \partial_{i-1}(f_i(x)) \oplus (\partial'_{i-1}(g_i(x)) + \partial'_{i-1}(f'_i(y)))$$
$$= f_{i-1}(\partial_{i-1}(x)) \oplus (g_{i-1}(\partial_{i-1}(x)) + f'_{i-1}(\partial'_{i-1}(y))$$
$$= \bar{f}_{i-1}(\partial_{i-1}(x) \oplus \partial'_{i-1}(y)) = (\bar{f}_{i-1} \circ (\partial_{i-1} \oplus \partial'_{i-1}))(x \oplus y).$$

Hence, $\bar{f} : C \oplus C' \to C \oplus C'$ is a chain map.

Step 2. Show that $\bar{f} : C \oplus C' \to C \oplus C'$ is a quasi-isomorphism.

Notice that the induced homomorphism $\bar{f}_{i*} : H_i(C) \oplus H_i(C') \to H_i(C) \oplus H_i(C')$ is defined by

$$\bar{f}_{i*}([x] \oplus [y]) = [f_i(x)] \oplus [g_i(x) + f'_i(y)]$$

for all $x \in Z_i(C)$ and $y \in Z_i(C')$ for each $i \in \{0, \dots, m\}$, where $[x] = x + B_i(C)$, $[f_i(x)] = f_i(x) + B_i(C)$, $[y] = y + B_i(C')$, and $[g_i(x) + f'_i(y)] = g_i(x) + f'_i(y) + B_i(C')$.

Let $i \in \{0, \dots, m\}$. We claim that $\bar{f}_{i*}$ is an isomorphism.

Suppose that $x_1, x_2 \in Z_i(C)$ and $y_1, y_2 \in Z_i(C')$ and $[f_i(x_1)] \oplus [g_i(x_1) + f'_i(y_1)] = [f_i(x_2)] \oplus [g_i(x_2) + f'_i(y_2)]$. Then $[f_i(x_1)] = [f_i(x_2)]$ and $[g_i(x_1) + f'_i(y_1)] = [g_i(x_2) + f'_i(y_2)]$. Since f_{i*} is 1-to-1, $[x_1] = [x_2]$, so $[g_i(x_1)] = [g_i(x_2)]$. Hence, $[f'_i(y_1)] = [f'_i(y_2)]$. Since f'_{i*} is 1-to-1, $[y_1] = [y_2]$. Therefore, $[x_1] \oplus [y_1] = [x_2] \oplus [y_2]$.

To show that $\bar{f}_{i*}$ is onto, let $x \in Z_i(C)$ and $y \in Z_i(C')$. Since f_{i*} is onto, there is $a \in Z_i(C)$ such that $[x] = [f_i(a)]$. Notice that $y - g_i(a) \in Z_i(C')$. Since f'_{i*} is onto, we can take $b \in Z_i(C')$ such that $[y - g_i(a)] = [f'_i(b)]$. Hence, $[y] = [g_i(a) + f'_i(b)]$. That is, $[x] \oplus [y] = [f_i(a)] \oplus [g_i(a) + f'_i(b)] = \bar{f}_{i*}([a] \oplus [b])$. Hence, $\bar{f} : C \oplus C' \to C \oplus C'$ is a quasi-isomorphism.

Step 3. Show that $\tau(\bar{f}) = \tau(f)\tau(f')$.

Since $\bar{f}$ is a quasi-isomorphism from $C \oplus C'$ to itself, by Theorem 3.20,

$$\tau(\bar{f}) = \prod_{i=0}^{m} [h_i \oplus h'_i / \bar{f}_{i*}(h_i \oplus h'_i)]^{(-1)^{i+1}} = \prod_{i=0}^{m} [h_i \oplus h'_i / f_{i*}(h_i) \oplus (g_{i*}(h_i) + f'_{i*}(h'_i))]^{(-1)^{i+1}}.$$

Note that for each $i \in \{0, \dots, m\}$,

$$(h_i \oplus h'_i / f_{i*}(h_i) \oplus (g_{i*}(h_i) + f'_{i*}(h'_i))) = (f_{i*}(h_i) \oplus (g_{i*}(h_i) + f'_{i*}(h'_i))/h_i \oplus h'_i)^{-1}$$

and

$$(f_{i*}(h_i) \oplus (g_{i*}(h_i) + f'_{i*}(h'_i))/h_i \oplus h'_i) = \begin{pmatrix} (f_{i*}(h_i)/h_i) & (g_{i*}(h_i)/h'_i) \\ 0 & (f'_{i*}(h'_i)/h'_i) \end{pmatrix}.$$

Hence,

$$[f_{i*}(h_i) \oplus (g_{i*}(h_i) + f'_{i*}(h'_i))/h_i \oplus h'_i] = [f_{i*}(h_i)/h_i][f'_{i*}(h'_i)/h'_i].$$

That is,

$$[h_i \oplus h'_i / f_{i*}(h_i) \oplus (g_{i*}(h_i) + f'_{i*}(h'_i))] = [h_i / f_{i*}(h_i)][h'_i / f'_{i*}(h'_i)].$$

Therefore,

$$\tau(\bar{f}) = \prod_{i=0}^{m} ([h_i / f_{i*}(h_i)][h'_i / f'_{i*}(h'_i)])^{(-1)^{i+1}} = \tau(f)\tau(f').$$

Obviously, if $g = 0$, then $\bar{f} = f \oplus f'$. By Lemma 3.6, $\tau(\bar{f}) = \pm\tau(f)\tau(f')$. Since f is a quasi-isomorphism from C to itself and f' is a quasi-isomorphism from C' to itself, we have $\tau(\bar{f}) = \tau(f)\tau(f')$.

This proves the theorem. $\square$

Now, let us consider the quotient of torsion of quasi-isomorphisms.

Theorem 3.22. *Let C and C' be based chain complexes of length m over a field F, and let $f : C \to C'$ and $g : C \to C'$ be quasi-isomorphisms. Then*

$$\frac{\tau(f)}{\tau(g)} = \prod_{i=0}^{m} [g_{i*}(h_i)/f_{i*}(h_i)]^{(-1)^{i+1}},$$

where h_i is a basis for $H_i(C)$ for each $i \in \{0, \ldots, m\}$.

Proof. Suppose that b_i, b_{i-1}, b_i', b_{i-1}', c_i, and c_i' are bases for $B_i(C)$, $B_{i-1}(C)$, $B_i(C')$, $B_{i-1}(C')$, C_i, and C_i', respectively, for each $i \in \{0, \ldots, m\}$. Then

$$\begin{aligned}
\frac{\tau(f)}{\tau(g)} &= \prod_{i=0}^{m} \left(\frac{[(b_i h_i) b_{i-1}/c_i]/[(b_i' f_{i*}(h_i)) b_{i-1}'/c_i']}{[(b_i h_i) b_{i-1}/c_i]/[(b_i' g_{i*}(h_i)) b_{i-1}'/c_i']} \right)^{(-1)^{i+1}} \\
&= \prod_{i=0}^{m} \left(\frac{[(b_i' g_{i*}(h_i)) b_{i-1}'/c_i']}{[(b_i' f_{i*}(h_i)) b_{i-1}'/c_i']} \right)^{(-1)^{i+1}} \\
&= \prod_{i=0}^{m} [(b_i' g_{i*}(h_i)) b_{i-1}'/(b_i' f_{i*}(h_i)) b_{i-1}']^{(-1)^{i+1}} = \prod_{i=0}^{m} [g_{i*}(h_i)/f_{i*}(h_i)]^{(-1)^{i+1}}.
\end{aligned}$$

$\square$

Note that $\prod_{i=0}^{m} [g_{i*}(h_i)/f_{i*}(h_i)]^{(-1)^{i+1}} = \prod_{i=0}^{m} [f_{i*}(h_i)/g_{i*}(h_i)]^{(-1)^{i}}$.

4. The torsion of dual map of a quasi-isomorphism

In this section, we introduce the duality theorem for torsion of a quasi-isomorphism. To prove this theorem, we need the following statements.

Proposition 4.1. *Let V and W be finite dimensional vector spaces over a field F, and let $f : V \to W$ be a linear transformation. Then if v and w be ordered bases for V and W, respectively, then $(f(v)/w) = (f^*(w^*)/v^*)^t$, where $f^* : W^* \to V^*$ is the dual map of f and $f(v)$ and $f^*(w^*)$ are the ordered images of v and w^* under f and f^*, respectively.*

Proof. Suppose that $\dim_F V = m$ and $\dim_F W = n$ and $v = (v_1, \ldots, v_m)$ and $w = (w_1, \ldots, w_n)$. Assume that the matrix $(f(v)/w)$ of f and the matrix $(f^*(w^*)/v^*)$ of

f^* are as follows.

$$(f(v)/w) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ and } (f^*(w^*)/v^*) = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{pmatrix}.$$

Let $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Then

$$f(v_i) = \sum_{k=1}^{n} a_{ik} w_k \quad \text{and} \quad f^*(w_j^*) = \sum_{k=1}^{m} b_{jk} v_k^*.$$

Hence, we have

$$(f^*(w_j^*))(v_i) = (w_j^* \circ f)(v_i) = w_j^* \left(\sum_{k=1}^{n} a_{ik} w_k \right) = \sum_{k=1}^{n} a_{ik} w_j^*(w_k) = a_{ij}$$

and

$$\left(\sum_{k=1}^{m} b_{jk} v_k^* \right)(v_i) = \sum_{k=1}^{m} b_{jk} v_k^*(v_i) = b_{ji}.$$

Therefore, $(f(v)/w) = (f^*(w^*)/v^*)^t$. That is, the matrix representations are the transpose of each other. $\square$

Proposition 4.2. *Let V_1, V_2, and V_3 be finite dimensional vector spaces over a field F. Then if $f : V_1 \to V_2$ and $g : V_2 \to V_3$ are linear transformations, then $(g \circ f)^* = f^* \circ g^*$. If $f = 0$, then $f^* = 0$.*

Proof. Let $h \in V_3^*$. Then $h : V_3 \to F$ is a linear functional and $(g \circ f)^*(h) = h \circ (g \circ f)$ and $(f^* \circ g^*)(h) = f^*(g^*(h)) = f^*(h \circ g) = (h \circ g) \circ f$. Hence, $(g \circ f)^* = f^* \circ g^*$. Also, if $f = 0$ and $h \in V_2^*$, then $f^*(h) = h \circ f = 0$, so $f^* = 0$. $\square$

Notice that, since the matrix representations $(f(v)/w)$ and $(f^*(w^*)/v^*)$ are the transpose of each other, they have the same rank. If V and W have the same dimension, the matrix representations have the same determinant.

To get a little simpler expression, from time to time, we use f^*b^* for $f^*(b^*)$ as follows.

Lemma 4.3. *Let V and W be finite dimensional vector spaces over a field F, and let $f : V \to W$ be a linear transformation, and let b be a basis for $\operatorname{Im} f$. Then if $\widetilde{b}$ is a lifting of b by f, then $f^*b^* = \widetilde{b}^*$ and f^*b^* is a basis for $\operatorname{Im} f^*$.*

Proof. Suppose that $\dim_F V = n$ and $\dim_F \operatorname{Im} f = r$ and k is an ordered basis for $\operatorname{Ker} f$. Let $k = (k_1, \ldots, k_{n-r})$ and $b = (b_1, \ldots, b_r)$. Then $\widetilde{b} = (\widetilde{b}_1, \ldots, \widetilde{b}_r)$ and $(k, \widetilde{b})$ is an ordered basis for V. Let $i \in \{1, \ldots, r\}$. Then for each $t \in \{1, \ldots, n-r\}$, $(f^*b_i^*)(k_t) = (b_i^* \circ f)(k_t) = b_i^*(0) = 0 = \widetilde{b}_i^*(k_t)$ and for each $j \in \{1, \ldots, r\}$, $(f^*b_i^*)(\widetilde{b}_j) =$

$(b_i^* \circ f)(\widetilde{b}_j) = b_i^*(b_j) = \delta_{ij} = \widetilde{b}_i^*(\widetilde{b}_j)$. Hence, $f^*b^* = \widetilde{b}^*$. Since f^*b^* is a linearly independent subset of $\operatorname{Im} f^*$ containing r elements and $\dim_F \operatorname{Im} f^* = \dim_F \operatorname{Im} f = r$, f^*b^* is a basis for $\operatorname{Im} f^*$. □

Lemma 4.4. *Let C be a chain complex over a field F, and let $\alpha \in C^i$. Then*

(1) $\alpha \in Z^i(C)$ if and only if $\alpha|_{B_i(C)} = 0$; (2) $\alpha \in B^i(C)$ if and only if $\alpha|_{Z_i(C)} = 0$, where $Z^i(C) = \operatorname{Ker}\{\delta^i : C^i \to C^{i+1}\}$ and $B^i(C) = \operatorname{Im}\{\delta^{i-1} : C^{i-1} \to C^i\}$. Also, $C^i = C_i^*$ and $\delta^i = \partial_i^*$ for each i.

Proof. (1) Let $\alpha \in Z^i(C)$. Then $\delta^i\alpha = 0$, so $\delta^i\alpha = \alpha \circ \partial_i = 0$. Hence, $(\alpha \circ \partial_i)(C_{i+1}) = \alpha(B_i(C)) = 0$. That is, $\alpha|_{B_i(C)} = 0$. Conversely, if $\alpha|_{B_i(C)} = 0$, then $\alpha(B_i(C)) = (\alpha \circ \partial_i)(C_{i+1}) = 0$. Hence, $\delta^i\alpha = \alpha \circ \partial_i = 0$. That is, $\alpha \in Z^i(C)$. (2) Let $\alpha \in B^i(C)$. Then $\alpha = \delta^{i-1}\beta$ for some $\beta \in C^{i-1}$. Hence, $\alpha = \beta \circ \partial_{i-1}$ and $\alpha(z) = (\beta \circ \partial_{i-1})(z) = \beta(0) = 0$ for all $z \in Z_i(C)$, so $\alpha|_{Z_i(C)} = 0$. Conversely, if $\alpha|_{Z_i(C)} = 0$, then $\operatorname{Ker}\partial_{i-1} = Z_i(C) \subseteq \operatorname{Ker}\alpha$. Hence, there is a unique homomorphism $\beta : B_{i-1}(C) \to F$ such that $\alpha = \beta \circ \partial_{i-1}$. Extend β to $\widetilde{\beta} : C_{i-1} \to F$. Therefore, $\alpha = \widetilde{\beta} \circ \partial_{i-1} = \delta^{i-1}\widetilde{\beta} \in B^i(C)$. □

Lemma 4.5. *The Universal Coefficient Theorem for a Field. If C is a chain complex of length m over a field F, then $H^i(C) \cong H_i(C)^*$ for each $i \in \{0, \dots, m\}$, where $H^i(C) = H_{m-i}(C^*)$ and $H_i(C)^* = \operatorname{Hom}_F(H_i(C), F)$.*

Proof. Let $i \in \{0, \dots, m\}$. Define a function $\Phi_i : Z^i(C) \to \operatorname{Hom}_F(H_i(C), F)$ by $\Phi_i(\alpha) = (\alpha|_{Z_i(C)})_*$ for each $\alpha \in Z^i(C)$. We claim that Φ_i is an epimorphism. Let $\alpha \in Z^i(C)$. Then $\alpha|_{B_i(C)} = 0$, so there is a unique homomorphism $(\alpha|_{Z_i(C)})_* : H_i(C) \to F$ such that $\alpha|_{Z_i(C)} = (\alpha|_{Z_i(C)})_* \circ \pi$, where $\pi : Z_i(C) \to H_i(C)$ is the canonical map. Hence, Φ_i is well-defined. Let $\alpha, \alpha' \in Z^i(C)$. Then $\alpha|_{B_i(C)} = \alpha'|_{B_i(C)} = 0$. For each $z \in Z_i(C)$, $((\alpha + \alpha')|_{Z_i(C)})_*([z]) = (\alpha + \alpha')(z) = \alpha(z) + \alpha'(z) = (\alpha|_{Z_i(C)})_*([z]) + (\alpha'|_{Z_i(C)})_*([z]) = ((\alpha|_{Z_i(C)})_* + (\alpha'|_{Z_i(C)})_*)([z])$. Also, for each $r \in F$, $((r\alpha)|_{Z_i(C)})_*([z]) = (r\alpha)(z) = r\alpha(z) = r(\alpha|_{Z_i(C)})_*([z]) = (r(\alpha|_{Z_i(C)})_*)([z])$, hence, Φ_i is a homomorphism. To show that Φ_i is onto, let $\beta \in \operatorname{Hom}_F(H_i(C), F)$. Then $\beta \circ \pi : Z_i(C) \to F$ is a homomorphism and $(\beta \circ \pi)|_{B_i(C)} = 0$. Extend $\beta \circ \pi$ to $\widetilde{\beta \circ \pi} : C_i \to F$. Since $\beta \circ \pi = (\beta \circ \pi)_* \circ \pi$ and π is onto, $\beta = (\beta \circ \pi)_*$. Hence, $\beta = \Phi_i(\widetilde{\beta \circ \pi})$ and $\widetilde{\beta \circ \pi} \in Z^i(C)$. That is, Φ_i is onto. Next, we show that $\Phi_i|_{B^i(C)} = 0$. Suppose that $\alpha \in B^i(C)$. Then $\alpha|_{Z_i(C)} = 0$ by Lemma 4.4, so $\Phi_i(\alpha) = (\alpha|_{Z_i(C)})_* = 0_* = 0$. Therefore, we have the induced homomorphism $\Phi_{i*} : H^i(C) \to \operatorname{Hom}_F(H_i(C), F)$ which is onto. By the universal coefficient theorem, we have the short exact sequence

$$0 \longrightarrow \operatorname{Ext}(H_{i-1}(C), F) \longrightarrow H^i(C) \longrightarrow \operatorname{Hom}_F(H_i(C), F) \longrightarrow 0\,.$$

Since F is a field, $\operatorname{Ext}(H_{i-1}(C), F) = 0$. Therefore, $H^i(C) \cong H_i(C)^*$. □

Lemma 4.6. *Let C and C' be chain complexes of length m over a field F such that C_i and C'_i are finite dimensional vector spaces for each $i \in \{0, \dots, m\}$, and let $f : C \to C'$ be a quasi-isomorphism. Then for each $i \in \{0, \dots, m\}$,*

(1) if h_i is a basis for $H_i(C)$, then $[\widetilde{h_i}^*]$ is a basis for $H^i(C)$;

(2) $(f_i^*)_*([f_i(\widetilde{h_i})^*]) = [\widetilde{h_i}^*]$.

Proof. Suppose that $\dim B_i(C) = r$ and $\dim H_i(C) = s$ and $\dim B_{i-1}(C) = t$ and $b_i = (b_{i1}, \dots, b_{ir})$ and $h_i = (h_{i1}, \dots, h_{is})$ and $b_{i-1} = (b_{i-11}, \dots, b_{i-1t})$. Notice that $(b_i, \widetilde{h_i}, \widetilde{b_{i-1}})$ is an ordered basis for C_i. Hence, $(b_i^*, \widetilde{h_i}^*, \widetilde{b_{i-1}}^*)$ is an ordered basis for C_i^*. In particular, $\widetilde{b_{i-1}}^*$ is an ordered basis for $B^i(C)$.

(1) For each $j \in \{1, \dots, s\}$ and $k \in \{1, \dots, r\}$, $\widetilde{h_{ij}}^*(b_{ik}) = 0$, that is, $\widetilde{h_{ij}}^*|_{B_i(C)} = 0$. Hence, by Lemma 4.4, $\widetilde{h_i}^* \subset Z^i(C)$. Suppose that $r_1[\widetilde{h_{i1}}^*] + \cdots + r_s[\widetilde{h_{is}}^*] = B^i(C)$. Then $r_1\widetilde{h_{i1}}^* + \cdots + r_s\widetilde{h_{is}}^* \in B^i(C)$. Hence, by Lemma 4.4, $(r_1\widetilde{h_{i1}}^* + \cdots + r_s\widetilde{h_{is}}^*)|_{Z_i(C)} = 0$. In particular, $(r_1\widetilde{h_{i1}}^* + \cdots + r_s\widetilde{h_{is}}^*)(\widetilde{h_{ij}}) = r_j = 0$ for each $j \in \{1, \dots, s\}$. Since $\dim H^i(C) = \dim H_i(C) = s$, $[\widetilde{h_i}^*]$ is a basis for $H^i(C)$.

(2) Note that $f_{i*}(h_i) = [f_i(\widetilde{h_i})]$, where $f_{i*}(h_i) = (f_{i*}(h_{i1}), \dots, f_{i*}(h_{is}))$ and $[f_i(\widetilde{h_i})] = ([f_i(\widetilde{h_{i1}})], \dots, [f_i(\widetilde{h_{is}})])$. Since $(f_i^*)_*([f_i(\widetilde{h_i})^*]) = [f_i^*(f_i(\widetilde{h_i})^*)]$, it suffices to show that $f_i^*(f_i(\widetilde{h_{ij}})^*) - \widetilde{h_{ij}}^* \in B^i(C)$ for each $j \in \{1, \dots, s\}$. We claim that $(f_i(\widetilde{h_{ij}})^* \circ f_i)|_{Z_i(C)} = \widetilde{h_{ij}}^*|_{Z_i(C)}$ for each $j \in \{1, \dots, s\}$. Remark that $(b_i, \widetilde{h_i})$ is a basis for $Z_i(C)$. Let $j \in \{1, \dots, s\}$ and $k \in \{1, \dots, r\}$ and $l \in \{1, \dots, s\}$. Then $\widetilde{h_{ij}}^*(b_{ik}) = 0$. since f_{i*} is an isomorphism, $f_i(\widetilde{h_{ij}}) \notin B_i(C')$, but $f_i(b_{ik}) \in B_i(C')$. Hence, $f_i(\widetilde{h_{ij}})^*(f_i(b_{ik})) = 0$. Also, $\widetilde{h_{ij}}^*(\widetilde{h_{il}}) = \delta_{jl}$ and $f_i(\widetilde{h_{ij}})^*(f_i(\widetilde{h_{il}})) = \delta_{jl}$ since f_{i*} is an isomorphism. Hence, $(f_i(\widetilde{h_i})^* \circ f_i)|_{Z_i(C)} = \widetilde{h_i}^*|_{Z_i(C)}$. Therefore, $(f_i^*)_*([f_i(\widetilde{h_i})^*]) = [\widetilde{h_i}^*]$. ☐

Now we introduce the torsion of dual map of a quasi-isomorphism. It turns out that the torsion of dual map of a quasi-isomorphism depends on the length of chain complexes which is domain or range of the quasi-isomorphism.

Theorem 4.7. *Let C and C' be based chain complexes of length m over a field F such that C_i and C'_i are finite dimensional vector spaces for each $i \in \{0, \dots, m\}$, and let $f : C \to C'$ be a quasi-isomorphism. Then $f^* : C'^* \to C^*$ is a quasi-isomorphism and $\tau(f^*) = \pm\, \tau(f)^{(-1)^m}$.*

Proof. For each $i \in \{0, \dots, m-1\}$, $\partial_i^* \circ f_i^* = (f_i \circ \partial_i)^* = (\partial'_i \circ f_{i+1})^* = f_{i+1}^* \circ \partial'^*_i$. Hence, $f^* : C'^* \to C^*$ is a chain map. Since $f_{i*} : H_i(C) \to H_i(C')$ is an isomorphism, $(f_{i*})^* : H_i(C')^* \to H_i(C)^*$ is an isomorphism. In the proof of the universal coefficient theorem for a field (Lemma 4.5), we defined the isomorphism $\Phi_{i*} : H^i(C) \to H_i(C)^*$. Define the isomorphism $\theta_{i*} : H^i(C') \to H_i(C')^*$ by the same argument as for C.

Now we claim that $(f_i^*)_* = \Phi_{i*}^{-1} \circ (f_{i*})^* \circ \theta_{i*}$. Let $\alpha \in Z^i(C')$. Then $[\alpha] \in H^i(C')$ and $\theta_{i*}([\alpha]) = (\alpha|_{Z_i(C')})_*$ and $(f_{i*})^*(\theta_{i*}([\alpha])) = (\alpha|_{Z_i(C')})_* \circ f_{i*} = (\alpha|_{Z_i(C')} \circ f_i)_* = ((\alpha \circ f_i)|_{Z_i(C)})_*$. Hence, $(\Phi_{i*}^{-1} \circ (f_{i*})^* \circ \theta_{i*})([\alpha]) = [\alpha \circ f_i] = [f_i^*(\alpha)] = (f_i^*)_*([\alpha])$. Therefore, $(f_i^*)_* : H^i(C') \to H^i(C)$ is an isomorphism, hence, $f^* : C'^* \to C^*$ is a quasi-isomorphism.

Now we consider the torsion $\tau(f^*)$ of the dual quasi-isomorphism f^* of f.

Note that for each $i \in \{0, \dots, m\}$, $(f^*)_i = (f_{m-i})^*$, $(C^*)_i = (C_{m-i})^*$, $(C'^*)_i = (C'_{m-i})^*$, $(\partial^*)_i = (\partial_{m-i-1})^*$, and $(\partial'^*)_i = (\partial'_{m-i-1})^*$.

Let us use the convention as follows. For each $i \in \{0, \dots, m\}$, $(f_i)^* = f_i^*$, $(C_i)^* = C_i^*$, $(C_i')^* = C'^*_i$, $(\partial_i)^* = \partial_i^*$, and $(\partial_i')^* = \partial'^*_i$. Then

$$
\begin{aligned}
\tau(f^*) &= \prod_{j=0}^{m} \left(\frac{[((b'^*)_j (h'^*)_j)(b'^*)_{j-1}/(c'^*)_j]}{[((b^*)_j (f^*)_{j*}((h'^*)_j))(b^*)_{j-1}/(c^*)_j]} \right)^{(-1)^{j+1}} \\
&= \prod_{i=0}^{m} \left(\frac{[((b'^*)_{m-i} (h'^*)_{m-i})(b'^*)_{m-i-1}/(c'^*)_{m-i}]}{[((b^*)_{m-i} (f^*)_{m-i*}((h'^*)_{m-i}))(b^*)_{m-i-1}/(c^*)_{m-i}]} \right)^{(-1)^{m-i+1}} \\
&= \prod_{i=0}^{m} \left(\frac{[(\partial'^*_{i-1} b'^*_{i-1},\, f_i(\widetilde{h}_i)^*,\, \widetilde{\partial'^*_i b'^*_i})/c'^*_i]}{[(\partial^*_{i-1} b^*_{i-1},\, f_i^* f_i(\widetilde{h}_i)^*,\, \widetilde{\partial^*_i b^*_i})/c^*_i]} \right)^{(-1)^{m-i+1}} && \text{By Lemma 3.4, 4.6.} \\
&= \prod_{i=0}^{m} \left(\frac{[(\widetilde{b'_{i-1}}^*,\, f_i(\widetilde{h}_i)^*,\, b'^*_i)/c'^*_i]}{[(\widetilde{b_{i-1}}^*,\, f_i^* f_i(\widetilde{h}_i)^*,\, b^*_i)/c^*_i]} \right)^{(-1)^{m-i+1}} && \text{By Lemma 4.3.} \\
&= \prod_{i=0}^{m} \left(\frac{[(\widetilde{b'_{i-1}}^*,\, f_i(\widetilde{h}_i)^*,\, b'^*_i)/c'^*_i]}{[(\widetilde{b_{i-1}}^*,\, \widetilde{h}_i^*,\, b^*_i)/c^*_i]} \right)^{(-1)^{m-i+1}} && \text{By Lemma 4.6.} \\
&= \pm \prod_{i=0}^{m} \left(\frac{[(b'^*_i,\, f_i(\widetilde{h}_i)^*,\, \widetilde{b'_{i-1}}^*)/c'^*_i]}{[(b^*_i,\, \widetilde{h}_i^*,\, \widetilde{b_{i-1}}^*)/c^*_i]} \right)^{(-1)^{m-i+1}} \\
&= \pm \prod_{i=0}^{m} \left(\frac{[(b'_i,\, f_i(\widetilde{h}_i),\, \widetilde{b'_{i-1}})/c'_i]^{-1}}{[(b_i,\, \widetilde{h}_i,\, \widetilde{b_{i-1}})/c_i]^{-1}} \right)^{(-1)^{m-i+1}} && \text{By Proposition 4.1} \\
&= \pm \prod_{i=0}^{m} \left(\frac{[(b_i,\, \widetilde{h}_i,\, \widetilde{b_{i-1}})/c_i]}{[(b'_i,\, f_i(\widetilde{h}_i),\, \widetilde{b'_{i-1}})/c'_i]} \right)^{(-1)^{i+1}(-1)^m} \\
&= \pm \left(\prod_{i=0}^{m} \left(\frac{[(b_i h_i) b_{i-1}/c_i]}{[(b'_i f_{i*}(h_i)) b'_{i-1}/c'_i]} \right)^{(-1)^{i+1}} \right)^{(-1)^m} && = \pm\, \tau(f)^{(-1)^m}.
\end{aligned}
$$

$\square$

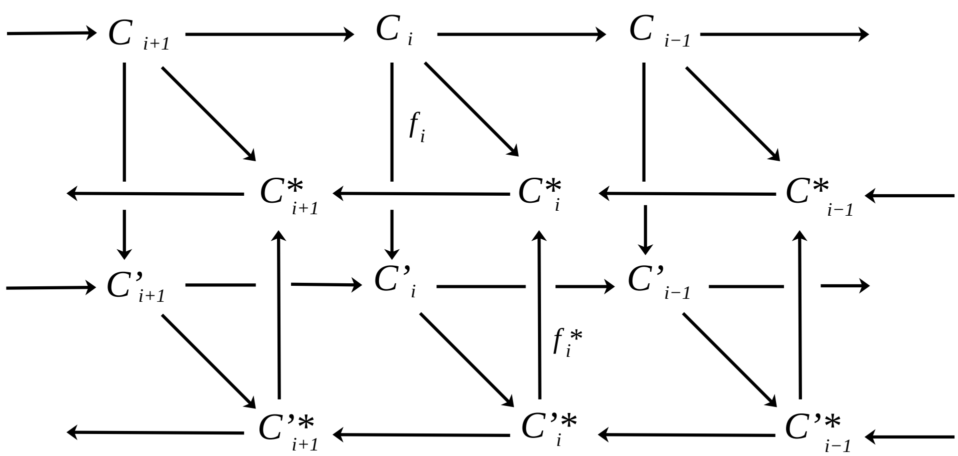

Figure 1. The diagram of a quasi-isomorphism and its dual map.

As the same idea that we determine the sign in Lemma 3.6, we have the exact sign for the torsion of dual map.

For each $i \in \{0, \ldots, m\}$, let

$$x_i = \dim_F B_i(C), \;\; x'_i = \dim_F B_i(C'), \;\; y_i = \dim_F H_i(C).$$

Then

$$\tau(f^*) = \frac{(-1)^{\sum_{i=0}^{m}(x'_i(x'_{i-1}+y_i)+x'_{i-1}y_i)}}{(-1)^{\sum_{i=0}^{m}(x_i(x_{i-1}+y_i)+x_{i-1}y_i)}}\,\tau(f)^{(-1)^m}.$$

Therefore,

$$\tau(f^*) = (-1)^{\sum_{i=0}^{m}[(x'_i(x'_{i-1}+y_i)+x'_{i-1}y_i)-(x_i(x_{i-1}+y_i)+x_{i-1}y_i)]}\,\tau(f)^{(-1)^m}.$$

5. Torsion of quasi-isomorphisms between free chain complexes

In this section, we generalize our torsion so that we define the torsion of a quasi-isomorphism between free chain complexes.

Suppose that R is an associative ring with $1 \neq 0$ which has invariant dimension property, that is, $m = n$ if and only if $R^m \cong R^n$, where m and n are nonnegative integers and R^m and R^n are direct sums of R. For each $n \in \mathbb{N}$, let $\mathrm{GL}(n, R)$ be the group of $n \times n$ invertible matrices over R, called the n-general linear group over R. We can identify each $A \in \mathrm{GL}(n, R)$ with the matrix

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(n+1, R)$$

so that we consider

$$\mathrm{GL}(1, R) \subset \mathrm{GL}(2, R) \subset \cdots.$$

$\mathrm{GL}(R) = \bigcup_{n\in\mathbb{N}} \mathrm{GL}(n, R)$ is called the infinite general linear group over R.

Notation 5.1.
$$K_1(R) = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)],$$
where $[\mathrm{GL}(R), \mathrm{GL}(R)]$ *is the commutator subgroup of* $\mathrm{GL}(R)$.

For the time being, let us assume that a ring R has $1 \neq 0$ and invariant dimension property, but need not be commutative. Consider a based chain complex C over R such that C_i is a based free R-module of finite rank for each i. We call such a chain complex a based free chain complex over R.

Suppose that C and C' are based free chain complexes of length m over R and $f : C \to C'$ is a quasi-isomorphism. If $B_i(C)$, $B_i(C')$, and $H_i(C)$ are free R-modules for all $i \in \{0, \ldots, m\}$, then we define the torsion $\tau(f)$ of $f : C \to C'$ by
$$\tau(f) = \left[\prod_{i=0}^{m} \left(\frac{((b_i h_i) b_{i-1}/c_i)}{((b'_i f_{i*}(h_i)) b'_{i-1}/c'_i)} \right)^{(-1)^{i+1}} \right] \in K_1(R),$$
where b_i, b_{i-1}, b'_i, b'_{i-1}, c_i, c'_i, and h_i are bases for $B_i(C)$, $B_{i-1}(C)$, $B_i(C')$, $B_{i-1}(C')$, C_i, C'_i, and $H_i(C)$, respectively, for each $i \in \{0, \ldots, m\}$. Also, $\frac{1}{A}$ means the inverse A^{-1} of A and $[A]$ means the abelianized class $A[\mathrm{GL}(R), \mathrm{GL}(R)]$ of A for $A \in \mathrm{GL}(R)$.

By the same idea as used in the proof of Lemma 3.4, we can show that $\tau(f)$ is well-defined. Note that if R is a commutative ring with 1, then the determinant $\det : K_1(R) \to R - \{0\}$ is a surjective group homomorphism. In particular, if R is a field, then $\det : K_1(R) \to R - \{0\}$ is an isomorphism. See [2].

Therefore, when R is a field, we can identify above definition with Definition 3.3 by this isomorphism.

In general, even though C is a free chain complex, its boundary and homology modules need not be free. We show that the torsion $\tau(f)$ of a quasi-isomorphism $f : C \to C'$ is defined if each homology module is a summand of a free module.

Lemma 5.2. *Let F be a free R-module, and let $m \in \mathbb{N}$ and $i \in \{0, \ldots, m-1\}$. Then if C is the chain complex of length m over R such that $C_i = C_{i+1} = F$ and $C_k = 0$ if $k \neq i$, $i+1$ and $\partial_i = I_F$ and $\partial_k = 0$ if $k \neq i$, then C is acyclic.*

Proof. We see that $\mathrm{Im}\,\partial_i = F$ and $\mathrm{Ker}\,\partial_{i-1} = F$. If $j \neq i$, then $\mathrm{Im}\,\partial_j = 0$ and $\mathrm{Ker}\,\partial_{j-1} = 0$. Hence, $\mathrm{Im}\,\partial_k = \mathrm{Ker}\,\partial_{k-1}$ for each $k \in \{0, \ldots, m\}$. That is, C is acyclic. $\square$

$$\cdots \longrightarrow 0 \longrightarrow F \xrightarrow{I_F} F \longrightarrow 0 \longrightarrow \cdots$$

Notation 5.3. *The chain complex in Lemma 5.2 is denoted by $C(F, i)$ for each $i \in \{0, \ldots, m-1\}$.*

Lemma 5.4. *If C and C' are acyclic chain complexes of length m over R, then $C \oplus C'$ is an acyclic chain complex of length m over R.*

Proof. Let $i \in \{0, \ldots, m\}$. Since C and C' are acyclic, $\operatorname{Im} \partial_i = \operatorname{Ker} \partial_{i-1}$ and $\operatorname{Im} \partial'_i = \operatorname{Ker} \partial'_{i-1}$. Hence, $\operatorname{Im} \partial_i \oplus \operatorname{Im} \partial'_i = \operatorname{Ker} \partial_{i-1} \oplus \operatorname{Ker} \partial'_{i-1}$. Note that $\operatorname{Im}(\partial_i \oplus \partial'_i) = \operatorname{Im} \partial_i \oplus \operatorname{Im} \partial'_i$ and $\operatorname{Ker}(\partial_{i-1} \oplus \partial'_{i-1}) = \operatorname{Ker} \partial_{i-1} \oplus \operatorname{Ker} \partial'_{i-1}$. Therefore, $C \oplus C'$ is acyclic. □

By Lemma 5.2 and 5.4, we have the following statement immediately which plays an important role for our generalization.

Lemma 5.5. *Let C be an acyclic based free chain complex of length $m \geq 1$ over R, and let F be a free based R-module. Then for each $i \in \{0, \ldots, m-1\}$, $C \oplus C(F_i, i)$ is an acyclic based free chain complex of length m over R. Furthermore, $C \oplus \bigoplus_{i=0}^{m-1} C(F_i, i)$ is also an acyclic based free chain complex of length m over R.*

Definition 5.6. An R-module M is said to be stably free if there is a free R-module F of finite rank such that $M \oplus F$ is free.

Note that zero R-module is stably free.

Lemma 5.7. *Let C be a based free chain complex of length m over R. Then if $H_i(C)$ is stably free for all $i \in \{0, \ldots, m\}$, then $Z_i(C)$ and $B_i(C)$ are stably free for all $i \in \{0, \ldots, m\}$.*

Proof. We prove this by induction. Suppose that $H_i(C)$ is stably free for all $i \in \{0, \ldots, m\}$, say, $H_i(C) \oplus F_i^H$ is free for some free R-module F_i^H. Remark that the direct sum of 2 short exact sequences

$$0 \longrightarrow B_i(C) \xrightarrow{\subseteq} Z_i(C) \xrightarrow{\pi} H_i(C) \longrightarrow 0$$

and

$$0 \longrightarrow 0 \longrightarrow F_i^H \xrightarrow{I_{F_i^H}} F_i^H \longrightarrow 0$$

is the short exact sequence

$$0 \longrightarrow B_i(C) \oplus 0 \longrightarrow Z_i(C) \oplus F_i^H \longrightarrow H_i(C) \oplus F_i^H \longrightarrow 0\,.$$

Since $Z_0(C) = C_0$, $Z_0(C)$ is free, so $Z_0(C)$ is stably free. Since $H_0(C) \oplus F_0^H$ is free, $Z_0(C) \oplus F_0^H = B_0(C) \oplus 0 \oplus H_0(C) \oplus F_0^H = B_0(C) \oplus H_0(C) \oplus F_0^H$, hence, $B_0(C)$ is stably free. Suppose that $B_{i-1}(C)$ is stably free for each $i \in \{1, \ldots, m\}$, say, $B_{i-1}(C) \oplus F_{i-1}^B$ is free for some free R-module F_{i-1}^B. To show that $B_i(C)$ is stably free, consider 3 short exact sequences

$$0 \longrightarrow Z_i(C) \xrightarrow{\subseteq} C_i \xrightarrow{\partial_{i-1}} B_{i-1}(C) \longrightarrow 0$$

and

$$0 \longrightarrow 0 \longrightarrow F_{i-1}^B \xrightarrow{I_{F_{i-1}^B}} F_{i-1}^B \longrightarrow 0$$

and

$$0 \longrightarrow F_i^H \xrightarrow{I_{F_i^H}} F_i^H \longrightarrow 0 \longrightarrow 0\,.$$

Then we have the short exact sequence

$$0 \longrightarrow Z_i(C) \oplus 0 \oplus F_i^H \longrightarrow C_i \oplus F_{i-1}^B \oplus F_i^H \longrightarrow B_{i-1}(C) \oplus F_{i-1}^B \oplus 0 \longrightarrow 0$$

which is the direct sum of those 3 short exact sequences. Hence,

$$\begin{aligned} C_i \oplus F_{i-1}^B \oplus F_i^H &= Z_i(C) \oplus F_i^H \oplus B_{i-1}(C) \oplus F_{i-1}^B \\ &= B_i(C) \oplus H_i(C) \oplus F_i^H \oplus B_{i-1}(C) \oplus F_{i-1}^B. \end{aligned}$$

Since $H_i(C) \oplus F_i^H \oplus B_{i-1}(C) \oplus F_{i-1}^B$ is free, $B_i(C)$ is stably free. Hence, $B_i(C)$ are stably free for all $i \in \{0, \ldots, m\}$.

Let $i \in \{0, \ldots, m\}$. Suppose that F_i^B and $B_i(C) \oplus F_i^B$ are free. Then we have the exact sequence

$$0 \longrightarrow B_i(C) \oplus 0 \oplus F_i^B \longrightarrow Z_i(C) \oplus F_i^H \oplus F_i^B \longrightarrow H_i(C) \oplus F_i^H \oplus 0 \longrightarrow 0.$$

Since $H_i(C) \oplus F_i^H$ is free, $Z_i(C) \oplus F_i^H \oplus F_i^B = B_i(C) \oplus F_i^B \oplus H_i(C) \oplus F_i^H$ and $Z_i(C) \oplus F_i^H \oplus F_i^B$ is free, so $Z_i(C)$ is stably free. This proves the lemma. □

Let C be a based free chain complex of length m over R. Suppose that $H_i(C)$ is stably free for all $i \in \{0, \ldots, m\}$. Then for each $i \in \{0, \ldots, m\}$, by Lemma 5.7, there are free R-modules F_i^B and F_i^H such that $B_i(C) \oplus F_i^B$ and $H_i(C) \oplus F_i^H$ are free. Notice that $C \oplus \bigoplus_{i=0}^{m-1} C(F_i^B, i) \oplus \bigoplus_{i=0}^{m} F_i^H$ is a free chain complex of length m over R whose boundary modules and homology modules are free. In particular,

$$H_i(C \oplus \bigoplus_{i=0}^{m-1} C(F_i^B, i) \oplus \bigoplus_{i=0}^{m} F_i^H) = H_i(C) \oplus F_i^H$$

for each $i \in \{0, \ldots, m\}$, where F_i^H is the free chain complex C' of length m over R such that the $C'_i = F_i^H$ and $C'_j = 0$ if $j \neq i$ and $\partial'_i = 0$ for all $i \in \{0, \ldots, m-1\}$.

Now, we generalize Definition 3.3 to the torsion of a quasi-isomorphism between free chain complexes.

Definition 5.8. Let C and C' be based free chain complexes of length m over R, and let $f : C \to C'$ be a quasi-isomorphism. Suppose that for each $i \in \{0, \ldots, m\}$, F_i^B, F_i^H, $F_i'^B$, and $F_i'^H$ are free R-modules such that $B_i(C) \oplus F_i^B$, $H_i(C) \oplus F_i^H$, $B_i(C') \oplus F_i'^B$, and $H_i(C') \oplus F_i'^H$ are free. Define the torsion $\tau(f)$ by

$$\tau(f) = \pm\, \tau(f \oplus S) \in K_1(R),$$

where S is the identity chain isomorphism from

$$\bigoplus_{i=0}^{m-1} C(F_i^B, i) \oplus \bigoplus_{i=0}^{m} F_i^H \oplus \bigoplus_{i=0}^{m-1} C(F_i'^B, i) \oplus \bigoplus_{i=0}^{m} F_i'^H$$

to
$$\bigoplus_{i=0}^{m-1} C(F_i^B, i) \oplus \bigoplus_{i=0}^{m} F_i^H \oplus \bigoplus_{i=0}^{m-1} C(F_i'^B, i) \oplus \bigoplus_{i=0}^{m} F_i'^H.$$
The identity chain isomorphism S is called a stabilizer of f.

Notice that, by definition, $\tau(S) = 1$ and $\tau(f \oplus S) = \pm\,\tau(S \oplus f)$.

Lemma 5.9. *Let C and C' be based free chain complexes of length m over R, and let $f : C \to C'$ be a quasi-isomorphism. Then if $H_i(C)$ and $H_i(C')$ are stably free for all $i \in \{0, \dots, m\}$ and S_1 and S_2 are stabilizers of f, then $\tau(f \oplus S_1) = \pm\,\tau(f \oplus S_2)$. That is, $\tau(f)$ is independent of the choice of stabilizer of f upto sign.*

Proof. Suppose that $f : C \to C'$ is a quasi-isomorphism between based free chain complexes of length m over R and $H_i(C)$ and $H_i(C')$ are stably free for all $i \in \{0, \dots, m\}$ and S_1 and S_2 are stabilizers of f. Then
$$\tau((f \oplus S_1) \oplus S_2) = \pm\,\tau(f \oplus S_1)\tau(S_2) = \pm\,\tau(f \oplus S_1)$$
and
$$\begin{aligned}\tau((f \oplus S_1) \oplus S_2) &= \pm\,\tau(S_2 \oplus (f \oplus S_1)) = \pm\,\tau((S_2 \oplus f) \oplus S_1)) \\ &= \pm\,\tau(S_2 \oplus f)\tau(S_1) = \pm\,\tau(S_2 \oplus f) = \pm\,\tau(f \oplus S_2).\end{aligned}$$
Therefore, $\tau(f \oplus S_1) = \pm\,\tau(f \oplus S_2)$. ☐

Next, we briefly introduce another extension of our torsion of quasi-isomorphisms which generalizes Turaev's Theorem [3] for the torsion of chain complexes whose rank of homology is zero.

Theorem 5.10. *Turaev* [3]*. Let R be a Noetherian unique factorization domain, and let C be a based free chain complex of length m over R such that* $\operatorname{rank} H_i(C) = 0$ *for all $i \in \{0, \dots, m\}$. Then $\widetilde{C} = \widetilde{R} \otimes_R C$ is a based acyclic chain complex of vector spaces over $\widetilde{R}$ and $\tau(C)$ is defined by $\tau(\widetilde{C})$. Furthermore,*
$$\tau(\widetilde{C}) = \prod_{i=0}^{m} \left(\operatorname{ord} H_i(C)\right)^{(-1)^{i+1}},$$
where $\operatorname{ord} H_i(C)$ *is the 0-th Alexander polynomial $\Delta_0(H_i(C))$ of $H_i(C)$ for each $i \in \{0, \dots, m\}$.*

Suppose that $f : C \to C'$ is a chain map which is not a necessarily quasi-isomorphism, but $\widetilde{f} = \mathrm{id} \otimes f : \widetilde{R} \otimes_R C \longrightarrow \widetilde{R} \otimes_R C'$ can be a quasi-isomorphism. In this case, we can define the torsion of f.

Definition 5.11. Let R be a Noetherian unique factorization domain, and let C and C' be based free chain complexes of length m over R, and let $f : C \to C'$ be a chain map such that $\widetilde{f} = \mathrm{id} \otimes f : \widetilde{R} \otimes_R C \longrightarrow \widetilde{R} \otimes_R C'$ is a quasi-isomorphism. Then the torsion $\tau(f)$ of f is defined by $\tau(f) = \tau(\widetilde{f})$.

This generalized torsion of chain maps has a possible application to link theory.

$$\begin{array}{ccc}
\widetilde{R} \otimes_R C_i & \xrightarrow{\mathrm{id}\otimes\partial_{i-1}} & \widetilde{R} \otimes_R C_{i-1} \\
{\scriptstyle \mathrm{id}\otimes f_i}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes f_{i-1}} \\
\widetilde{R} \otimes_R C'_i & \xrightarrow{\mathrm{id}\otimes\partial'_{i-1}} & \widetilde{R} \otimes_R C'_{i-1}
\end{array}$$

Corollary 5.12. *Let R be a Noetherian unique factorization domain, and let C and C' be based free chain complexes of length m over R such that $\operatorname{rank} H_i(C) = 0$ and $\operatorname{rank} H_i(C') = 0$ for all $i \in \{0, \ldots, m\}$, and let $f : C \to C'$ be a chain map such that $\widetilde{f} = \mathrm{id} \otimes f : \widetilde{R} \otimes_R C \longrightarrow \widetilde{R} \otimes_R C'$ is a quasi-isomorphism. Then*

$$\tau(f) = \tau(\widetilde{f}) = \frac{\tau(\widetilde{C})}{\tau(\widetilde{C'})} = \prod_{i=0}^{m} \left(\frac{\operatorname{ord} H_i(C)}{\operatorname{ord} H_i(C')} \right)^{(-1)^{i+1}},$$

where $\widetilde{C} = \widetilde{R} \otimes_R C$ and $\widetilde{C'} = \widetilde{R} \otimes_R C'$.

6. Examples of quasi-isomorphisms

In this section, we introduce a few concrete examples of torsion of quasi-isomorphism. Notice that in all these examples, the quasi-isomorphisms are from a chain complex to itself. So by Theorem 3.20, their torsion can be calculated from their actions on homology. Nevertheless, we want to show the calculation from definition so that the reader may get familiar with the construction.

Note that a vector space C can be regarded as a chain complex $0 \to C \to 0$ with length 0 and a bijective linear transformation $f : C \to C$ is a quasi-isomorphism. In this case, the torsion of f is exactly same as $\det f$ (Theorem 3.20).

$$\begin{array}{ccccccc}
0 & \xrightarrow{\partial_1} & C_1 & \xrightarrow{\partial_0} & C_0 & \xrightarrow{\partial_{-1}} & 0 \\
\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & \downarrow \\
0 & \xrightarrow{\partial_1} & C_1 & \xrightarrow{\partial_0} & C_0 & \xrightarrow{\partial_{-1}} & 0
\end{array}$$

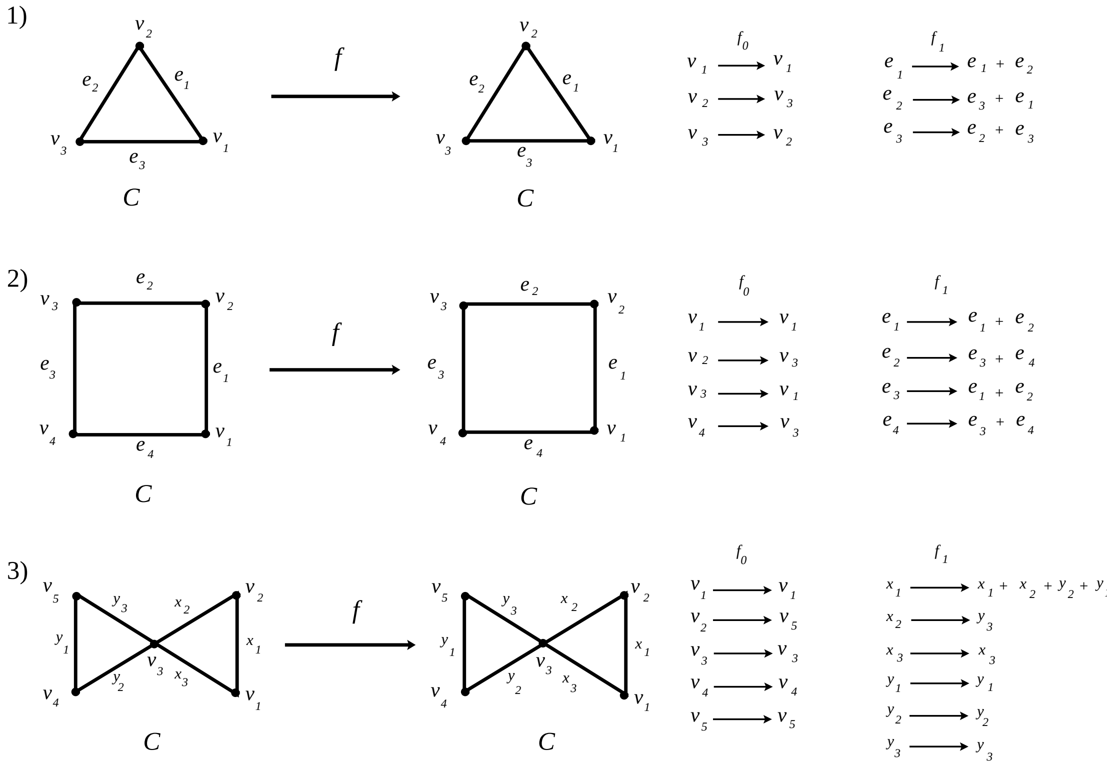

Figure 2. Examples of quasi-isomorphisms.

The chain complexes in the examples are chain complexes over the real field $\mathbb{R}$.

Example 1. We compute the torsion of $f : C \to C$ as 1) in Figure 2 by definition. Suppose that $C_0 = \langle v_1, v_2, v_3\rangle$, $C_1 = \langle e_1, e_2, e_3\rangle$, $c_0 = (v_1, v_2, v_3)$, and $c_1 = (e_1, e_2, e_3)$ and the 0-th boundary map $\partial_0 : C_1 \to C_0$ is defined by $\partial_0(e_1) = v_2 - v_1$, $\partial_0(e_2) = v_3 - v_2$, and $\partial_0(e_3) = v_1 - v_3$. Then we can easily show that f is a chain map. Since $B_0(C) = \mathrm{sp}(v_2 - v_1, v_3 - v_2, v_1 - v_3) = \mathrm{sp}(v_2 - v_1, v_3 - v_2)$, we can take $b_0 = (v_2 - v_1, v_3 - v_2)$. Also, $H_0(C) = C_0/B_0(C)$. To take a basis h_0 for $H_0(C)$, we need a vector in C_0 not contained in $B_0(C)$. For this, we take an orthogonal vector $v_1+v_2+v_3$ to $B_0(C)$, so we take $h_0 = (v_1 + v_2 + v_3 + B_0(C))$. Since $f_{0*}(v_1 + v_2 + v_3 + B_0(C)) = v_1 + v_3 + v_2 + B_0(C)$, f_{0*} is the identity map. Suppose that $\partial_0(r_1e_1 + r_2e_2 + r_3e_3) = 0$. Then $r_1(v_2 - v_1) + r_2(v_3 - v_2) + r_3(v_1 - v_3) = 0$. Hence, $r_1 = r_2 = r_3$ and we have $Z_0(C) = \langle e_1 + e_2 + e_3\rangle$. Since $B_0(C) = 0$, $H_0(C) = Z_0(C)$. Take $h_1 = (e_1 + e_2 + e_3 + 0)$. Since $f_{1*}(e_1 + e_2 + e_3 + 0) = 2(e_1 + e_2 + e_3) + 0$ and $2(e_1 + e_2 + e_3) \notin B_1(C) = 0$, f_{1*} is an isomorphism.

Now, we compute the torsion $\tau(f)$ of f. Notice that $b_{-1} = b_1 = \emptyset$. Since $(b_0h_0)b_{-1} = (v_2 - v_1, v_3 - v_2, v_1 + v_2 + v_3)$, we have $[(b_0h_0)b_{-1}/c_0] = \det \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 3$. Also,

$(b_0 f_{0*}(h_0))b_{-1} = (v_2 - v_1, v_3 - v_2, v_1 + v_3 + v_2)$, hence, $[(b_0 f_{0*}(h_0))b_{-1}/c_0] = 3$. Similarly, since $(b_1 h_1)b_0 = (e_1 + e_2 + e_3, e_1, e_2)$ and $(b_1 f_{1*}(h_1))b_0 = (2(e_1 + e_2 + e_3), e_1, e_2)$, we have $[(b_1 h_1)b_0/c_1] = 1$ and $[(b_1 f_{1*}(h_1))b_0/c_1] = 2$. Therefore,

$$\tau(f) = \left(\frac{[(b_0 h_0)b_{-1}/c_0]}{[(b_0 f_{0*}(h_0))b_{-1}/c_0]}\right)^{-1}\left(\frac{[(b_1 h_1)b_0/c_1]}{[(b_1 f_{1*}(h_1))b_0/c_1]}\right) = \left(\frac{3}{3}\right)^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\,.$$

Example 2. We use Theorem 3.20 to compute the torsion of $f : C \to C$ as 2) in Figure 2. Suppose that $C_0 = \langle v_1, v_2, v_3, v_4\rangle$, $C_1 = \langle e_1, e_2, e_3, e_4\rangle$, $c_0 = (v_1, v_2, v_3, v_4)$, and $c_1 = (e_1, e_2, e_3, e_4)$ and the 0-th boundary map $\partial_0 : C_1 \to C_0$ is defined by $\partial_0(e_1) = v_2 - v_1$, $\partial_0(e_2) = v_3 - v_2$, $\partial_0(e_3) = v_4 - v_3$, and $\partial_0(e_4) = v_1 - v_4$. Then f is a chain map. Since $B_0(C) = \mathrm{sp}(v_2 - v_1, v_3 - v_2, v_4 - v_3, v_1 - v_4) = \mathrm{sp}(v_2 - v_1, v_3 - v_2, v_4 - v_3)$, we can take $b_0 = (v_2 - v_1, v_3 - v_2, v_4 - v_3)$. Also, $H_0(C) = C_0/B_0(C)$. To take a basis h_0 for $H_0(C)$, we need a vector in C_0 not contained in $B_0(C)$. As in Example 1, we can take $h_0 = (v_1 + v_2 + v_3 + v_4 + B_0(C))$. Since $f_{0*}(v_1 + v_2 + v_3 + v_4 + B_0(C)) = v_1 + v_3 + v_1 + v_3 + B_0(C)$ and $(v_1 + v_2 + v_3 + v_4) - (v_1 + v_3 + v_1 + v_3) = (v_2 - v_1) + (v_4 - v_3) \in B_0(C)$, f_{0*} is the identity map. Hence, $\det f_{0*} = 1$. As in Example 1, we have $Z_0(C) = \langle e_1 + e_2 + e_3 + e_4\rangle$. Let us take $h_1 = (e_1 + e_2 + e_3 + e_4 + 0)$. Then $f_{1*}(e_1 + e_2 + e_3 + e_4 + 0) = 2(e_1 + e_2 + e_3 + e_4) + 0$. Since $B_1(C) = 0$, $2(e_1 + e_2 + e_3 + e_4) \notin B_1(C)$. Hence, f_{1*} is an isomorphism and $\det f_{1*} = 2$. Therefore, by Theorem 3.20,

$$\tau(f) = \left(\frac{1}{\det f_{0*}}\right)^{-1}\left(\frac{1}{\det f_{1*}}\right) = \left(\frac{1}{1}\right)^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\,.$$

First two examples are about 2-fold covering maps. Finally, we try to get the torsion of a little bit more complicated quasi-isomorphism.

Example 3. We also use Theorem 3.20 for the torsion of $f : C \to C$ as 3) in Figure 2. Suppose that $C_0 = \langle v_1, v_2, v_3, v_4, v_5\rangle$, $C_1 = \langle x_1, x_2, x_3, y_1, y_2, y_3\rangle$, $c_0 = (v_1, v_2, v_3, v_4, v_5)$, and $c_1 = (x_1, x_2, x_3, y_1, y_2, y_3)$ and the 0-th boundary map $\partial_0 : C_1 \to C_0$ is defined by $\partial_0(x_1) = v_2 - v_1$, $\partial_0(x_2) = v_3 - v_2$, $\partial_0(x_3) = v_1 - v_3$, $\partial_0(y_1) = v_5 - v_4$, $\partial_0(y_2) = v_4 - v_3$, and $\partial_0(y_3) = v_3 - v_5$. Then f is a chain map. Since $B_0(C) = \mathrm{sp}(v_2 - v_1, v_3 - v_2, v_1 - v_3, v_5 - v_4, v_4 - v_3, v_3 - v_5)$, we can take $b_0 = (v_2 - v_1, v_3 - v_2, v_5 - v_4, v_4 - v_3)$. Also, $H_0(C) = C_0/B_0(C)$. As in Example 1, we take $h_0 = (v_1 + v_2 + v_3 + v_4 + v_5 + B_0(C))$. Since $f_{0*}(v_1 + v_2 + v_3 + v_4 + v_5 + B_0(C)) = v_1 + v_3 + v_4 + 2v_5 + B_0(C)$ and $(v_1 + v_3 + v_4 + 2v_5) - (v_1 + v_2 + v_3 + v_4 + v_5) = v_5 - v_2 = (v_5 - v_4) + (v_4 - v_3) + (v_3 - v_2) \in B_0(C)$, f_{0*} is the identity map. Hence, $\det f_{0*} = 1$.

Suppose that $\partial_0(r_1 x_1 + r_2 x_2 + r_3 x_3 + s_1 y_1 + s_2 y_2 + s_3 y_3) = 0$. Then $r_1(v_2 - v_1) + r_2(v_3 - v_2) + r_3(v_1 - v_3) + s_1(v_5 - v_4) + s_2(v_4 - v_3) + s_3(v_3 - v_5) = 0$ and we have $r_1 = r_2 = r_3$ and $s_1 = s_2 = s_3$. Hence, $Z_0(C) = \langle x_1 + x_2 + x_3, y_1 + y_2 + y_3\rangle$. Since $B_0(C) = 0$, $H_0(C) = Z_0(C)$. Take $h_1 = (x_1 + x_2 + x_3 + 0, y_1 + y_2 + y_3 + 0)$. Since $f_{1*}(x_1 + x_2 + x_3 + 0) = (x_1 + x_2 + x_3 + 0) + (y_1 + y_2 + y_3 + 0)$ and $f_{1*}(y_1 + y_2 + y_3 + 0) =$

$y_1 + y_2 + y_3 + 0$. Hence, the matrix representation of f_{1*} for h_1 is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\det f_{1*} = 1 \neq 0$. Therefore f_{1*} is an isomorphism and, by Theorem 3.20, we have

$$\tau(f) = \left(\frac{1}{\det f_{0*}}\right)^{-1} \left(\frac{1}{\det f_{1*}}\right) = \left(\frac{1}{1}\right)^{-1} \left(\frac{1}{1}\right) = 1.$$

References

[1] Allen Hatcher, Algebraic Topology, Cambridge University Press & Co., Cambridge, 2002.

[2] Vladimir G. Turaev, Introduction to Combinatorial Torsions, Birkhauser & Co., Basel, 2001.

[3] Vladimir G. Turaev, *Reidemeister torsion in knot theory*, Russian Math. Surveys, 41(1986) 119–182.

[4] John W. Milnor, *Infinite cyclic coverings*, Conference on the Topology of Manifolds, (J.G. Hocking, ed), Prindle, Weber and Schmidt, Boston, 1968, pp. 115–133.

Department of Mathematics, University of California, Riverside, CA 92521

E-mail address: `xl@math.ucr.edu, jwchung@alumni.ucr.edu`